Mean-Variance Efficiency and the Capital Asset Pricing Model

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Abstract

I provide a simple proof of Roll's (1977) result that mean-variance efficiency and the capital asset pricing model equation are mathematically equivalent.

Proposition

Let x_p denote any mean-variance efficient portfolio with expected return $E[R_p]$, except for the minimum variance portfolio. For all assets i, the Capital Asset Pricing Model equation holds:

$$E[R_i] - E[R_{zp}] = \beta_i(E[R_p] - E[R_{zp}])$$

where $E[R_{zp}]$ the unique mean-variance efficient portfolio satisfying $Cov(R_p, R_{zp}) = 0$. Furthermore, any pair of portfolios (x_p, x_{zp}) that satisfy the above conditions are necessarily both mean-variance efficient.

Proof

Let $\Sigma = (\sigma_{i,j})_{N\times N}$ denote the matrix of asset covariances. Suppose x_p is efficient; that is, it is the solution to the optimization problem

$$x_p = \arg\ \min \frac{1}{2}x^T\Sigma x \text{ subject to } x_p = ER_p,\ \ x_p^T\mathbf{1} = 1.$$

The necessary and sufficient first order conditions for a solution are:

$$\begin{aligned} \Sigma x_p &= \lambda\mu + \kappa\mathbf{1} \\ ER_p &= x_p^{'}\mu \\ 1 &= x_p^{'}\mathbf{1}. \end{aligned} \tag{1}$$

Solving these first order conditions for x_p yields

$$x_p = (\lambda\Sigma^{-1}\mu + \kappa\Sigma^{-1}1),\ \ x_p'\mu = E[R_p]$$

where λ and κ are

$$\lambda = \frac{1}{D}(CE[R_p] - A),\ \ \gamma = \frac{1}{D}\left(B - AE[R_p]\right). \tag{2}$$

and where the scalars (A, B, C, D) are defined as

$$A = \mu^{'}\Sigma^{-1}\mathbf{1}, \;\; B = \mu^{'}\Sigma^{-1}\mu, \;\; C = \mathbf{1}\Sigma^{-1}\mathbf{1}, \;\; D = BC - A^2.$$

By definition of covariance,

$$Cov(R_p, R_q) = x_p'\Sigma x_q.$$

for all portfolios x_p and x_q. Suppose x_p is efficient. Using the first order (1) it follows that

$$\begin{aligned} Cov(R_p, R_q) &= (\lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}\mathbf{1})'\Sigma x_q \\ &= \lambda\mu'\Sigma^{-1}\Sigma x_q + \gamma 1'\Sigma^{-1}x_q \\ &= \lambda\mu^{'}x_q + \gamma 1' x_q \\ &= \lambda E[R_q] + \gamma. \end{aligned}$$

Substituting in values of λ and γ yields the expression

$$Cov(R_p, R_q) = \frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}.$$

Let x_{zp} denote any portfolio satisfying $Cov(R_p, R_{zp}) = 0$. Such a portfolio always exists and lies on the frontier. The expected return $E[R_{zp}]$ is given by rearranging the covariance expression:

$$E[R_{zp}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}}.$$

The variance of R_p is $Cov(R_p, R_p)$. Dividing the above by $Var(R_p)$ yields

$$\frac{Cov(R_p, R_q)}{Var(R_p)} = \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}$$

Isolating $E[R_q]$ and setting $\frac{Cov(R_p,R_q)}{Var(R_p)} = \beta_q$ yields,

$$\begin{aligned} E[R_q] &= \frac{\beta_q Var(R_p) - \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + E[R_{zp}]. \end{aligned}$$

The first term in this epression can be further decomposed:

$$\frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} = \beta_q \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)}$$

$$
\begin{aligned}
&= \beta_q \left(E[R_p] - \frac{A}{C} + \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}} \right) \\
&= \beta_q (E[R_p] - E[R_{zp}])
\end{aligned}
$$

Substituting this this back into the expression $E[R_q]$ yields the CAPM equation

$$E[R_q] = \beta_q (E[R_p] - E[R_{zp}]) + E[R_{zp}].$$

We have now proven that mean-variance efficiency of x_p implies the CAPM equation.

To prove that the CAPM equation implies mean-variance efficiency of x_p, the previous algebraic manipulations can be reversed to obtain the necessary and sufficient first order conditions for mean-variance efficiency. By sufficiency, it immediately follows x_p is mean-variance efficient.

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