Simulating Stochastic Differential Equations and Applications in Pricing Asian-type Options

Matthew Pollard*

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Abstract

This report serves as an introduction to the related topics of simulating diffusions and option pricing. Specifically, it considers diffusions that can be specified by stochastic differential equations by $dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t$, and pricing Asian options, a type of path-dependent options where no general pricing formula is known. Two numeric approximations of X_t , the Euler and the Milstein, are derived through application of Itô's lemma. It shows that the Euler has an error order $O(\Delta t)$ and the Milstein has error order $O(\Delta t)^{3/2}$. Geometric brownian motion and the Ornstein-Ulhbeck process are simulated using the Euler method and solved analytically using the method of reduction. The Euler approximation is used with Monte Carlo methods to estimate the price of Asian options. The price estimates obtained from Monte Carlo simulation are compared to the analytic Black-Scholes formula.

^{*}Department of Statistics; University of California, Berkeley.

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1 Introduction

This paper introduces two intimately related subjects, solving stochastic differential equations and pricing financial options. Specifically, it looks at numerical methods to simulate Itô processes specified by the equation

$$dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t$$

and applies this simulation technique to price Asian options, where unlike for European and American options, no general analytic pricing formula is known.

Pricing options is one of the central problems in Finance. In 1973, Fisher Black and Myron Scholes published the now celebrated the Black-Scholes model and formula. Their paper gave simple formulae for the true value of a European call option under the assumption that prices follow geometric Brownian processes. Since then, the options world has seen explosive growth in several directions. An entire academic industry has developed around developing new pricing formulae, proposing new models and developing numerical techniques. And, a jungle of new "exotic" options types has grown, with species like "knock-back," "Bermudean," "Digital," "Quanto," and "Rainbow" to accommodate for every bet or hedge position imaginable. Asian options belong to this class of exotics, and no general formulae is known to price them. In absence of this, numerical methods are the only alternative.

Several numerical methods exist to price options. One method is to reduce the option pricing problem to solving partial differential equation. Indeed, this was originally how the Black-Scholes formula was discovered, through solving a PDE. Exotic options, however, rarely have tractable PDE solutions. Instead, generic numerical methods, such as finite differences, can be applied.

A simpler and more attractive method is Monte Carlo estimation. This is the technique presented in this paper, and for Asian options, involves simulating the whole sample path of an Itô process. The principle is to simulate many realizations of stock prices and calculate the terminal option value for each. The sample average of the realized values converges to the expectation of the payoff distribution by the Law of Large Numbers. Under suitable transformations, this expectation is the fair value of an option.

Simulating the Itô processes used in Monte Carlo is achieved through approximation. Two approximations, the Euler and the Milstein, are derived through application of Itô's lemma. Numerical theory is is complemented with an introduction to solving SDEs analytically. Theorems for the existence and uniqueness are presented, and the method of reduction is used to solve two important SDEs: the geometric Brownian motion, which is used in the Black-Scholes model, and the Ornstein-Uhlenbeck process, used in the Vaseik interest-rate model.

The paper is structured as follows. Section 2 is a short introduction to Itô processes and stochastic calculus. The Itô integral is defined, and the basic tools of Itô's lemma and isometry, with proof, are presented. Section 3 proves approximation order formulae for the Euler and Milstein method. Section 4 presents basic theory of existance and uniqueness of SDE solutions and how to solve SDEs that are reducible. Section 5 looks at two examples of SDEs, geometric brownian motion and the Ornstein-Uhlenbeck process, and uses reducibility to solve the SDEs analytically. Section 6 presents basic options theory, proves the Black-Scholes formula, introduces the Monte Carlo pricing method and shows how to price Asian options through Monte Carlo simulation.

The following notation will be used throughout the report:

μ	instantaneous drift
σ	instantaneous variance
S_t	asset price at time t
S_0	initial asset price
T	expiration date of option contract
K	strike price of option contract
r	annual riskless interest rate
P_T	option value at maturity
$f(\cdot)$	exercise payoff function
$(x)^+$	$\max(x,0)$
Q	risk neutral measure

2 Itô Processes and Calculus

Many stochastic models in physics, economics and finance reduce to a simple diffusion differential equation of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \tag{1}$$

where W_t is the standard Brownian motion process, $\mu(x,t)$ and $\sigma(x,t)$ are deterministic and differentiable functions. Such processes are called Itô, after mathematician Itô who laid the groundwork for stochastic calculus. Equivalently, the integral equation for the above diffusion is given by

$$X_{t} = X_{0} + \int_{0}^{t} \mu(X_{s}, s)ds + \int_{0}^{t} \sigma(X_{t}, s)dW_{s}$$
⁽²⁾

The process (X_t) is Markovian in the sense that at any point t in time, $(X_s)_{s\geq t}$ depends only on X_t and not $(X_s)_{s\leq t}$. The stochastic differential dX is interpreted as a limit of increment $X_{t+h} - X_t$, which has mean $\mu(X_t, t)h$ and variance $\sigma^2(X_t, t)h$. The stochastic integrals are defined by the limit

$$\int_0^t f(X_s, s) dW_s = \lim_{n \to \infty} \int h_n(X_s, s) dW_t$$
$$= \lim_{n \to \infty} \sum_{i=1}^n h_n(X_s, t) (W_{s_i} - W_{s_{i-1}})$$

where $h_n(X_t, s)$ is a simple process¹ that converges almost everywhere to $f(X_t, t)$ and $W_{s_i} - W_{s_{i+1}}$ are independent $\mathcal{N}(0, s_i - s_{i-1})$ increments.

The fundamental method of numerically simulating the process (X_t) is to discretize the time interval [0, t] into $\{0, \Delta t, 2\Delta t, ..., (\frac{t}{\Delta t} - 1)\Delta t, t\}$. Letting $t_i = i\Delta x$, equation (2) implies

$$X_{t_{i+1}} = X_{t_i} + \int_{t_i}^{t_{i+1}} \mu(X_s, s) ds + \int_{t_i}^{t_{i+1}} \sigma(X_t, s) dW_s$$
(3)

Our goal is to find good approximations (3). To do this, we need several two important tools of stochastic calculus, the celebrated Itô lemma and the Itô Isometry formula. Itô lemma is a generalization of the Fundamental Theorem of Calculus to stochastic integrals and is extremely important in studying them. Itô Isometry is highly useful in evaluating integrals.

Itô's lemma Let W_t be a Brownian motion process and X_t an Itô process with $dX_t = \mu(x,t)dt + \sigma(x,t)dW_t$. Let $Y_t = f(X_t,t)$ and suppose f_x, f_{xx} and f_t are all continuous functions. Then

$$dY_t = \left(\mu(X_t, t)f_x + f_t + \frac{1}{2}\sigma^2(X_t, t)f_{xx}\right)dt + \sigma(X_t, t)f_x dW_t$$

where $f_x = \frac{\partial f(x,t)}{\partial x} | \{x = X_t\}.$

I refer readers to to Shreve, page 168 for a rigourous proof. Now for Itô Isometry.

Itô Isometry For any measurable process X_s ,

$$\mathbb{E}\left[\left(\int_0^t X_s dW_s\right)^2\right] = \int_0^t \mathbb{E}[X_s^2] ds$$

Proof Let $(t_k)_k$ be a partition of [0, t]. Approximate the Itô integral by the sum

$$\sum_{0 < t_k < t} X_{t_k} \Delta W_k$$

Since each ΔW_k is independent,

$$\mathbb{E}\left(\sum_{0 < t_k < t} X_{t_k} \Delta W_k\right)^2 = \mathbb{E}\sum_{0 < t_k < t} X_{t_k}^2 (\Delta W_k)^2 + \mathbb{E}\sum_{t_j < t_k} X_{t_j} X_{t_k} \Delta W_j \Delta W_k$$
$$= \sum_{0 < t_k < t} X_{t_k}^2 \Delta t_k$$

Taking the limit of the mesh of $(t_k)_k$ to zero gives the result.

¹For the sum to converge almost surely, a technical condition for $h_n(X_t, t)$ is required: $\int E|h_n|^2 ds < \infty$. Loosely speaking, this is the stochastic equivalent of L^p integrability for functions.

Example: Integrating $\int_0^t W_s dW_s$ Let us evaluate

$$I = \int_0^t W_s dW_s.$$

If the Itô integral obeyed the Fundamental Theorem of Calculus, then the value of the integral would be $I = \frac{W_s^2}{2} \Big|_0^t = \frac{W_t^2}{2}$. Unsurprisingly, however, this wrong. We will use Itô to evaluate it.

Let $f(W_t, t) = W_t^2$. Then $f_x = 2x$, $f_{xx} = 2$, $f_t = 0$ and $\mu = 0$, $\sigma^2 = 1$ for all X_t and t. Applying Itô gives

$$d(W_t^2) = dt + 2W_t dW_t$$

Integrating on both sides gives

$$\int_0^t d(W_s^2) = t + 2 \int_0^t W_s dW_s$$

 $\int_0^t d(W_s^2) = W_s^2$, so rearranging gives

$$\int_0^t W_s dW_s = \frac{W_t^2 - t}{2}.$$

This shows that the Itô calculus is fundamentally different than ordinary calculus.

Now we're ready to play with approximating (1).

3 Simulating Stochastic Differential Equations

Suppose X_t is an satisfies a diffusion equation of the form (1). Let $t_i = i\Delta t$, then we have for discrete points $\{X_{t_i}\}_{i=1}^n$,

$$X_{t_{i+1}} = X_{t_i} + \int_{t_i}^{t_{i+1}} \mu(X_s, s) ds + \int_{t_i}^{t_{i+1}} \sigma(X_t, s) dW_s.$$

The challenge is to estimate the continuous time integrals $\int_{t_i}^{t_{i+1}} \mu(X_s, s)$ and $\int_{t_i}^{t_{i+1}} \sigma(X_t, s) dW_s$. Fortunately, this can be done through a straight forward, albeit algebraically tedious, application of Itô's lemma. Two approximations, the Euler and the Milstein approximation are derived in this section. The first has error $O_p(\Delta t)$ and the second has error $O_p(\Delta t)^{3/2}$.

Theorem 1 (Euler and Milstein)

$$X_{t_{i+1}} = X_{t_i} + \mu(X_{t_i}, t_i)\Delta t + \sigma(X_{t_i}, t)\Delta W_t + O_p(\Delta t) \quad \text{(Euler)}$$

$$X_{t_{i+1}} = X_{t_i} + \mu(X_{t_i}, t_i)\Delta t + \sigma(X_{t_i}, t)\Delta W_t + \frac{1}{2}\sigma(X_{t_i}, t_i)\frac{\partial}{\partial x}\sigma(X_{t_i}, t_i)[(\Delta W_t)^2 - \Delta t] + O_p(\Delta t)^{3/2} \quad \text{(Milstein)}$$

Proof Define the partial differential operators L^0 and L^1 that operate on f by

$$L^0 f = a \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t}, \qquad L^1 f = \sigma \frac{\partial f}{\partial x}.$$

We can express Itô's lemma by L^0 and L^1 by

$$df(X_t, t) = L^0 f dt + L^1 f dW_t$$

We'll apply this to functions μ and σ . Assume the appropriate derivatives exists. Integrating over $[t_i, s]$ yields

$$\mu(X_s, s) = \mu(X_{t_i}, t_i) + \int_{t_i}^s L^0 \mu(X_u, u) du + \int_{t_i}^s L^1 \mu(X_u, u) dW_u$$

$$\sigma(X_s, s) = \sigma(X_{t_i}, t_i) + \int_{t_i}^s L^0 \sigma(X_u, u) du + \int_{t_i}^s L^1 \sigma(X_u, u) dW_u$$

Now substitute μ and σ into the integrands of

$$X_{t_{i+1}} = X_{t_i} + \int_{t_i}^{t_{i+1}} \mu(X_s, s) ds + \int_{t_i}^{t_{i+1}} \sigma(X_t, s) dW_s$$

Working on the first integral, $\int_{t_i}^{t_{i+1}} \mu(X_s,s) ds$, we have

$$\begin{split} \int_{t_i}^{t_{i+1}} \mu(X_s, s) ds &= \int_{t_i}^{t_{i+1}} \left(\mu(X_{t_i}, t_i) + \int_{t_i}^s L^0 \mu(X_u, u) du + \int_{t_i}^s L^1 \mu(X_u, u) dW_u \right) ds \\ &\approx \quad \mu(X_{t_i}, t_i) \Delta t + L^0 \mu(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^s du ds + L^1 \mu(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^s dW_u ds \end{split}$$

The first term $\mu(X_{t_i}, t_i)\Delta t$ is a first order approximation to the desired integral, and the rest is a lower order correction that we can regard as a error term. The second term, $L^0\mu(X_{t_i}, t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^s duds$ is $O_p(\Delta t)^2$ because

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^s du ds = \frac{\Delta t^2}{2}$$

and $L^0\mu(X_{t_i}, t_i)$ is bounded. The third term $L^1\mu(X_{t_i}, t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^s dW_u ds$ is $O_p(\Delta t)^{3/2}$ since

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^s dW_u ds = \int_{t_i}^{t_{i+1}} (t_{i+1} - u) dW_u \sim \mathcal{N}(0, \frac{\{\Delta t\}^3}{3})$$

Thus the integral can be written as $3^{-1/2} (\Delta t)^{3/2} Z$, where Z is a standard normal random variable, and clearly this is $O_p(\Delta t)^{3/2}$.

Similarly for the second integral,

$$\int_{t_i}^{t_{i+1}} \sigma(X_s, s) dW_s = \int_{t_i}^{t_{i+1}} \left(\sigma(X_{t_i}, t_i) + \int_{t_i}^s L^0 \sigma(X_u, u) du + \int_{t_i}^s L^1 \sigma(X_u, u) dW_u \right) ds$$

$$\approx \quad \sigma(X_{t_i}, t_i) \Delta W_t + L^0 \sigma(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^s du dW_t + L^1 \sigma(X_{t_i}, t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^s dW_u dW_s$$
$$= \quad \sigma(X_{t_i}, t_i) \Delta W_t + \frac{1}{2} \sigma(X_{t_i}, t_i) \frac{\partial}{\partial x} \sigma(X_{t_i}, t_i) [(\Delta W_t)^2 - \Delta t] + O_p (\Delta t)^{3/2}$$

since

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^s dW_u dW_s = \frac{1}{2} [(\Delta W_t)^2 - \Delta, t],$$
$$L^0 \sigma(X_u, u) = \sigma(X_{t_i}, t_i) + O_p(\Delta t)^{1/2}, \ L^1 \sigma(X_u, u) = \sigma(X_u, u) \frac{\partial}{\partial x} \sigma(X_u, u), \text{ and}$$
$$\int_{t_i}^{t_{i+1}} \int_{t_i}^s du dW_s = O_p(\Delta t)^{3/2}.$$

Putting these terms together, we have the Milstein approximation to increment ΔX_t given by

$$\Delta X_t = \mu(X_{t_i}, t_i)\Delta t + \sigma(X_{t_i}, t_i)\Delta W_t + \frac{1}{2}\sigma(X_{t_i}, t_i)\frac{\partial}{\partial x}\sigma(X_{t_i}, t_i)[(\Delta W_t)^2 - \Delta t] + O_p(\Delta t)^{3/2}$$

4 Solving Stochastic Differential Equations

While we are now armed with two methods to simulate Itô processes, simulations do not constitute *solutions* in a stochastic setting. Rather, they are sample paths of solutions. Knowing a closed form solution is useful, and this section presents basic theory of analytically solving SDEs. Specifically, for a given differential equation $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$, $X_0 = x_0$, we seek a solution of the form $X_t = f(t, W_t)$ that, if it exists, is unique.

Clearly, the solution depends on how "nice" μ and σ are. In ordinary differential equation theory, Lipschitz continuity is a sufficient condition. The following theorem states gives us that this is also the case with SDEs.

Theorem 2 (Existence and Uniqueness) If $\mu(x,t)$ and $\sigma(x,t)$ are continuous on and if for some finite K,

- 1. $|\mu(x,t) \mu(y,t)| + |\sigma(x,t) \sigma(y,t)| \le K|x-y|$ (Lipshitz Continuity)
- 2. $|\mu(x,t)| + |\sigma(x,t)| \le K(1+|x|)$ (boundedness)

then for any $T \ge 0$, the SDE has a unique solution $(X_s)_{0 \le t \le T}$. The solution satisfies

$$\mathbb{E}\{\sup_{0 < t < T} X_s^2\} < \infty.$$

4.1 Reducible Stochastic Differential Equations

There is a rich class of SDEs that are solvable through the *method of reduction*. The idea is find an invertible transformation $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $Y_t \equiv f(X_t,t)$ satisfies the SDE

$$dY_t = r(t)dt + q(t)dW_t, \ Y_0 = f(0, x_0) = 0.$$

Solving this SDE is straightforward. Integrate both sides with respect to t,

$$Y_t = \int_0^t r(s)ds + \int_0^t q(s)dW_s$$

and given $f^{-1}(t, x)$ exists, we have $X_t = f^{-1}(Y_t, t)$. The following theorem gives conditions on which μ and σ give reducible SDEs and the partial differential equation that f solves.

Theorem 3 (Method of Reduction) If the coefficient functions $\mu(x,t)$ and $\sigma(x,t)$ satisfy the partial differential equation

$$\frac{\partial}{\partial x} \left\{ \frac{1}{\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial t} - \sigma(t,x) \frac{\partial}{\partial x} \left(\frac{\mu(x,t)}{\sigma(x,t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(t,x) \right) \right\} = 0$$

There exists a transformation $Y_t = f(X_t, t)$ that transforms the stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \ X_0 = x_0$$

to the stochastic differential equation

$$dY_t = r(t)dt + q(t)dW_t, \ Y_0 = f(0, x_0) = 0.$$

The coefficients r(t) and $\sigma(t)$ of the transformed equation are determined by the system of partial differential equations

$$\begin{aligned} \frac{dq(t)}{dt} &= q(t) \left\{ \frac{1}{\sigma(x,t)} \frac{\partial \sigma(x,t)}{\partial t} - \sigma(t,x) \frac{\partial}{\partial x} \left(\frac{\mu(x,t)}{\sigma(x,t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(t,x) \right) \right\} \\ r(t) &= q(t) \left\{ \frac{\mu(x,t)}{\sigma(x,t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(t,x) \right\} + \frac{\partial}{\partial t} \left\{ q(t) \int_{x_0}^x \frac{dy}{\sigma(y,t)} dy \right\} \end{aligned}$$

and the transformation, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, by the system of partial differential equations

$$\begin{array}{lcl} \frac{\partial f}{\partial x}(x,t) & = & \frac{q(t)}{\sigma(x,t)} \\ \\ \frac{\partial f}{\partial t}(x,t) & = & r(t) - q(t) \left\{ \frac{\mu(x,t)}{\sigma(x,t)} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(t,x) \right\}. \end{array}$$

The next section provides two examples of solving SDEs by the method of reduction.

5 Examples and Simulations

This section presents two famous examples of diffusions: Geometric Brownian motion and the Ornstein-Uhlenbeck process. The SDE is solved analytically, and numerical approximations are used to simulate and plot sample paths of the solution.

5.1 Geometric Brownian Motion

Let X_t be a solution to

$$dX_t = rX_t dt + \sigma X_t dW_t. \tag{4}$$

Then X_t is called a geometric Brownian motion. Heuristically, if $\sigma \approx 0$, we have $X_t \approx rX_t dt$ and thus X_t increases exponentially at rate approximately r. As X_t increases, the volatility σX_t linearly increases.

This SDE has been widely used to model stock prices. Recall the formula for asset return over s to t: $r_{s,t} \equiv \log \frac{P_t}{P_s}$; solving gives $P_t = P_s e^{r_{s,t}}$, and letting s converge to t yields $\frac{dP_t}{dt} = r_t P_t$, where r_t is the instantaneous interest rate $r_t = \overline{r} + \varepsilon_t$, ε_t is a iid noise process with variance σ , then the variation in P_t due to ε_t for small Δt is $P_t e^{\varepsilon_t}$. The variance grows at rate proportional to P_t .

The celebrated Black-Scholes model, which we'll see later, assumes stock prices follow Geometric Brownian motion processes. Figure (1) shows a simulated Geometric Brownian motion price path and the S&P 500 stock index over the last 50 years. Underneath each are the realized returns. At first glance, geometric Brownian motion seems to model *prices* well. Indeed, it is a good first order approximation - prices grow exponentially and variance increases proportionally to price level. However, looking at *returns* shows clearly that the model significantly strays from empirical relality. There are far more extreme points in real data than normality predicts. Also, returns seem to fluctuate wildly in some periods and remain tame in others.

Now to solve this SDE. The solution is

$$X_t = X_0 e^{(r - \sigma^2/2)t + \sigma W_t}.$$

Proof (Method of reduction) We have $\mu(x,t) = rx$ and $\sigma(x,t) = \sigma x$. This SDE is reducible since

$$\frac{\partial}{\partial x} \left\{ \frac{1}{\sigma x} \frac{\partial \sigma x}{\partial t} - \sigma x \frac{\partial}{\partial x} \left(\frac{rx}{\sigma x} - \frac{1}{2} \frac{\partial \sigma x}{\partial x} \right) \right\} = \frac{\partial}{\partial x} \left\{ 0 - \sigma x \frac{1}{2} \frac{\partial^2 \sigma x}{\partial x^2} \right\} = 0.$$

The transformed SDE functions r(t) and q(t) solve

$$\begin{aligned} \frac{dq(t)}{dt} &= 0 \Rightarrow q(t) = Q \\ r(t) &= Q\left\{\frac{r}{\sigma} - \frac{1}{2}\sigma\right\} + \frac{\partial}{\partial t}\left\{q(t)\int_{x_0}^x \frac{1}{\sigma y}dy\right\} = Q\left\{\frac{r}{\sigma} - \frac{1}{2}\sigma\right\} + \frac{C}{\sigma}\frac{\partial}{\partial t}\left\{\log(x) - \log(x_0)\right\} \\ &= Q\left\{\frac{r}{\sigma} - \frac{1}{2}\sigma\right\} \equiv R \end{aligned}$$

Thus, the transformed SDE reads

$$dY_t = Rdt + QdW_t, \ Y_0 = f(0, x_0) = 0$$

whose solution is $Y_t = rt + QW_t$. All that is left is to find the transformation f(x, t). f solves



Figure 1: Left: Simulated Prices and Returns of Geometric Brownian Motion, Right: Prices and Returns of S&P 500 index from 1950 to 2006.

the PDF system

$$\frac{\partial f}{\partial x}(x,t) = \frac{Q}{\sigma x}, \quad \frac{\partial f}{\partial t}(x,t) = R - R = 0$$

Thus $f(x,t) = \frac{Q}{\sigma}\log(x) + c_0$. By the boundary condition $f(x_0,0) = 0, c_0 = \frac{Q}{\sigma}\log(x_0)$ and consequently

$$f(x,t) = \frac{Q}{\sigma} \log(\frac{x}{x_0})$$

The inverse exists: $f^{-1}(x,t) = x_0 e^{\frac{\sigma}{Q}x}$. Inserting the transformed process Y_t yields

$$X_t = f^{-1}(Y_t, t) = x_0 e^{\frac{\sigma}{Q}(Rt + QW_t)}$$

Now, $\frac{\sigma}{Q}R = r - \frac{1}{2}\sigma^2$, thus the final form of our solution is

$$X_t = x_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

An easy exercise shows that $\mathbb{E}X_t = x_0 e^{rt}$ and $Var X_t = x_0^2 e^{2tr} (e^{2t\sigma^2} - e^{t\sigma^2})$. A simpler



Figure 2: Geometric Brownian Motion given by SDE solution (black) and Euler approximation (red)

and more intuitive method to solve the geometric Brownian motion SDE using coefficient matching is presented in the appendix. This method, however, only works for an extremely limited class of linear SDEs.

Figure to compare a sample geometric Brownian motion given by the solution

5.2 Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is a simple mean-reverting diffusion. It has dynamics specified by

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

where μ is real, $\theta > 0$ and $\sigma > 0$. μ is best considered a long run tendency point. Where $X_t = \mu$, the process has zero drift and is locally martingale. The process does *not* converge to X_t since the dW_t term still causes the process to wander randomly about μ . θ determines the strength of reversion and, consequently, the time-lagged correlation structure. The model has been widely used in academic literature to model both physical and economic processes. It has modelled temperature fluctuation (Keilson and Ross) and interest rates (Vasicek) among others.

The solution to this SDE is

$$X_t = X_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW_s$$

Proof (method of reduction) We have $\mu(x,t) = \theta(\mu - x)$ and $\sigma(x,t) = \sigma$. This SDE is reducible since

$$\frac{\partial}{\partial x} \left\{ 0 - \sigma \frac{\partial}{\partial x} \left(\frac{\theta(\mu - x)}{\sigma} - 0 \right) \right\} = 0$$

The transformed SDE functions r(t) and q(t) solve

$$\begin{aligned} \frac{dq(t)}{dt} &= \theta q(t) \Rightarrow q(t) = Q e^{\theta t}, \\ r(t) &= \frac{\theta(\mu - x)q(t)}{\sigma} + \frac{(x - x_0)q'(t)}{\sigma} \\ &= \frac{\theta(\mu - x_0)q(t)}{\sigma} = \frac{\theta Q(\mu - x_0)e^{\theta t}}{\sigma} \end{aligned}$$

Thus, the transformed SDE reads

$$dY_t = \frac{\theta Q(\mu - x_0)e^{\theta t}}{\sigma} dt + Qe^{\theta t} dW_t, \ Y_0 = f(0, x_0) = 0$$

whose solution is

$$Y_t = \frac{Q(\mu - x_0)(e^{\theta t} - 1)}{\sigma} + \frac{Q(e^{\theta t} - 1)}{\theta} W_t.$$

All that is left is to find the transformation f(x, t). f solves the PDF system

$$\frac{\partial f}{\partial x}(x,t) = \frac{Qe^{\theta t}}{\sigma}, \quad \frac{\partial f}{\partial t}(x,t) = \frac{Q\theta e^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x_0,0) + \frac{(x-x_0)Qe^{\theta t}}{\sigma} = 0, \quad \Rightarrow f(x,t) = f(x$$

and inverting to $x = f^{-1}(y, t)$ yields

$$x = x_0 + \frac{\sigma}{Qe^{\theta t}} \{ y - f(0, x_0) \}$$

$$\Rightarrow X_t = x_0 + \frac{\sigma}{Qe^{\theta t}} \{ Y_t - Y_0 \} = x_0 + \frac{\sigma}{Qe^{\theta t}} \int_0^t dY_s$$

$$= x_0 + \frac{\sigma}{Qe^{\theta t}} \left\{ \int_0^t \frac{\theta Q(\mu - x_0)e^{\theta s}}{\sigma} ds + \int_0^t Qe^{\theta s} dW_s \right\}$$

$$= x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta (s-t)} dW_s$$

The expectation of X_t is easily see to be $\mathbb{E}X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$ since $\mathbb{E}\int_0^t e^{\theta(s-t)} dW_s = 0$. The covariance $\operatorname{cov}(X_s, X_t)$ is harder. Let $s \wedge t = \min(s, t)$. We can use Itô isometry to calculate the covariance function by

$$\begin{aligned} \operatorname{cov}(X_s, X_t) &= & \mathbb{E}[(X_s - \mathbb{E}[X_s])(X_t - \mathbb{E}[X_t])] \\ &= & \mathbb{E}[\int_0^s \sigma e^{\theta(u-s)} dW_u \int_0^t \sigma e^{\theta(v-t)} dW_v] \\ &= & \sigma^2 e^{-\theta(s+t)} \mathbb{E}[\int_0^s \sigma e^{\theta u} dW_u \int_0^t \sigma e^{\theta v} dW_v] \end{aligned}$$



Figure 3: Simulated Ornstein-Uhlenbeck process with parameters $\mu = 0$, $\theta = 1$, $\sigma = 0.3$, n = 1000; green: $X_0 = 2$, red: $X_0 = 0$, blue: $X_0 = -2$.

$$= \frac{\sigma^2}{2\theta} e^{-\theta(s+t)} (e-1).$$

The variance is

$$Var(X_t) = \frac{\sigma^2}{2\theta} (1 - e^{-2\theta t})$$

which is bounded by $\frac{\sigma^2}{2\theta}$. Thus the process admits a stationary probability distribution, in contrast to Brownian motion.

6 Options Theory

The problem of pricing options is one of the central problems in finance. Nearly all financial instruments can be grouped as either assets (shares, bonds) or derivatives of assets, which are functions of shares and bond prices. Derivatives can be further decomposed to either forward contracts and options. Pricing forward contracts is relatively easy compared to option pricing. The latter has bread a jungle of dense theory, heavily borrowing from stochastic analysis, integration theory, partial differential equation analysis and statistical physics. Option pricing is as complicated as rocket science.

6.1 Options Basics

A *call* option is a contract that gives the holder the right (but not the obligation) to buy a fixed amount of an asset at a specified time in future for an already agreed price, the *strike price*, from the seller (or writer of the option). The opposite option is a *put* option. By buying it, the holder receives the right to sell a fixed amount of asset to writer for the strike price. Here the writer is obliged to buy the asset while the holder may decide on selling or

not.

A European call option on one share of stock offers the buyer of the option the right to by this share at time t = T for strike price K = 0 which is fixed at time t = 0. If the share price S_t at t = T exceeds the strike price, then the holder can exercise the option and buy the share at price $K < S_T$. He can then immediately sell the share at the market price S_T and make a profit of $S_T - K$. If, however, we have $S_T \leq K$, then the option holder will not exercise the option since he can buy the stock at an equal or lower price than K. In this case, the profit from holding the option is zero. Therefore, the profit V from holding a European option is given by

$$V_c = (S_T - K)^+$$

where $(x)^+$ denotes $\max(S_T - K, 0)$. Similarly, the payoff for a put option is

$$V_p = (K - S_T)^+.$$

In general, let f(t, K) denote the payoff function. Several alternative option type payoffs are given below.

American call :
$$(S_{\tau} - K)^+, \ \tau \in [0, T]$$

Asian call : $\left(S_T - T^{-1} \int_0^T S_t \ dt\right)^+$
Fixed-strike average call : $\left(T^{-1} \int_0^T S_t \ dt - K\right)^+$

The above payoffs are the intrinsic value or price of the option at maturity. What is more interesting is the there price *before* maturity, say at time t = 0. when the option is written. The fair price², P, of the option is given by the expectation

$$P = \mathbb{E}_Q\{e^{-rT}f(S_T)\}$$

where r is the risk free interest rate and the expectation is over the probability measure Q. What is Q? It is *not* the physical distribution of price S_T that is specified by the model for $(S_t)_t$. Instead, it is the "risk-neutral" measure. Under Q, each asset traded in the economy has a fair price that traders can agree upon, irrelevant to their subjective beliefs about the direction of future prices. Indeed, under Q, each asset earns an average return rate equal to r. Every asset option has fair price at t = 0 equal to the Q expectation of $f(S_T)$ times by the discounting factor e^{-rT} . Furthermore, at these prices, the market does not admit arbitrage. Conversely, no arbitrage implies the existence of Q. This is the Fundamental Theorem of Arbitrage Pricing and forms the bedrock of all option pricing theory.

²This is the price that rules out arbitrage opportunities in the market. Under the Black-Scholes model, the price is unique. Under more complicated models, for example, the trinomial market model, an interval of fair prices exists. Also, some models, such as Mandlebrot's random multi-fractal model, arbitrage opportunities exist.

6.2 The Black-Scholes Formula

A formula for European option prices under some limiting assumptions exists: the celebrated Black-Scholes formula. The main limiting assumptions ("The Black-Scholes Model") are as follows. First, the market has a risk free asset with return $r_t > 0$ and at least one stock, i.e. the market is *complete*. Second, the price of the stock evolves by the geometric Brownian motion

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t. \tag{5}$$

The volatility σ is constant, however μ_t and r_t are not necessarily. For simplicity however, take $\mu_t = \mu$ and $r_t = r$. Now for the result.

The Black-Scholes Formula The fair prices of European call and put options with strike price K > 0 and maturity T are

$$P_c = S_0 \cdot \Phi\{d_1\} - K \cdot e^{-rT} \Phi\{d_2\}, \quad P_p = K \cdot e^{-rT} \Phi\{-d_2\} - S_0 \cdot \Phi\{d_1\}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where Φ is the standard normal cumulative distribution function.

Proof Assume the market admits no arbitrage. Then by the fundamental theorem of arbitrage, there exists a risk neutral measure. For *all* assets in the economy, under this risk neutral measure, $\mu = r$ and

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Let $S_T^* = e^{-rT}S_T$, the discounted stock price at time T. This has log-normal distribution with mean $\log S_0 - \sigma^2 T/2$ and variance $\sigma^2 T$ and $P = \mathbb{E}(S_T^* - Ke^{-rt})^+$. A routine but lengthy integration by parts yields the formula. See [1] for the calculation.

6.3 Monte Carlo Pricing

Monte Carlo simulation is a powerful numeric technique to evaluate the payoff expectation. The concept is very intuitive: simulate many price paths of the underlying asset with respect to a given model, then calculate the payoff of each simulation and approximate $\mathbb{E}X$ by the sample mean \bar{X} .

It is easy to implement, but computationally intensive. It is especially useful in pricing options where producing a formula for the price expectation is extremely difficult or impossible. Such cases frequently arise when pricing non-European style options, for example, the with so-called Asian and Knock-back options. This is also true when relaxing the Black-Scholes assumptions of Gaussian returns with constant volatility. The simplest estimation algorithm is as follows.

Basic Monte Carlo Algorithm



Figure 4: Monte Carlo Simulation in practice. Simulate thousands of realizations of the risk-neutral price path and for each, calculate the discounted terminal payoff and estimate the expectation by the sample mean.

- 1. for k = 1, 2, ..., N
 - (a) generate price path $S^{(k)} = \{S_0, S_{\Delta t}, ..., S_T\}^{(k)}$
 - (b) compute payoff $V_k = f(S^{(k)})$
- 2. let $\widehat{P}_T = e^{-rT}\overline{V} = e^{-rT}\frac{\sum_{k=1}^N V_k}{N}$

This algorithm is implemented in R function, **MCsim**. A plot of several price paths generated by the algorithm is shown in figure (4).

The basic Monte Carlo estimator, while unbiased, Ursula has a large variance. For options where each Monte Carlo point estimate contains many sampling operations, such as with Asian options as we'll see later, finding a way to reduce variances attractive. A large literature is devoted to Monte Carlo variance reduction method. See [2, 3] and [1] for an overview.

Two attractive techniques specifically applicable to option pricing are stratified sampling and importance sampling. One reason why the variance of the Monte Carlo estimator is high is the large interval in which the payoff function is zero. Naturally we would prefer to concentrate the sample in the region where the payoff function is positive and, where it is more variable, use larger sample sizes. Stratified sampling works by tilting the distribution of S_T so that more sample payoffs are positive, then un-tiling by applying a discounting factor to ensure the estimate remains unbiased. Importance sampling tilts the underlying distribution so that important aspects, such as tails, are sampled more frequently. The simulation outputs are re-weighted to ensure the estimate is unbiased, and these weights are given by the Radon-Nikodym derivate of the underlying distribution with respect to the simulation distribution.

6.3.1 Example: Pricing a European Call

Suppose we wish to price a European call with parameters $K = 100, S_0 = 100, r = 0.1, \sigma = 0.1$ and T = 1 year. Recall that under the Black-Scholes model,

$$S_T = S_0 \exp([r - \sigma^2/2]T + \sigma W_T)$$

and $V = (S_T - K)^+$. This payoff is independent of the paths $(S_t)_{t \leq T}$. Rather, all that matters is S_T and the distribution of this is given above. We can price the option with n = 10000 simulations in R using the code:

```
ST<-100*exp((0.1-0.1^2/2)*1+0.1*rnorm(10000))
Discounted<-exp(-T*r)*pmax(z*(sim-K),0)}
mean(Discounted)
[1] 10.30117</pre>
```

Or, using the MCsim function:

```
MCsim(S=100,K=100,r=0.1,sigma=0.1,n=10000,type='ce')
[1] 10.30096
```

These calculuations took less than a second on a P4 1.7 ghz laptop. The value using the Black-Scholes formula is

```
BSEuro(S=100,K=100,r=0.1,sigma=0.1,type='ce')
[1] 10.30815
```

The difference between the estimates is 0.7 cents, which is less than the common minimum price unit ("tick") of 1 cent.

6.4 Asian Options

Asian options are derivatives with payoff which depend on the average of the underlying stock price. Specifically, Asian options have payoffs for call and puts given by

$$V_c = (S_T - \overline{S}_T)^+, \quad V_p = (\overline{S} - S_T)^+,$$

where $\overline{S} = T^{-1} \int_0^T S_t dt$, the continuous time arithmetic average of S_t . In discrete time, $\overline{S} = \frac{1}{T} \sum_{t=1}^T S_t$. Thus, unlike European, Asian options are *path-dependent* - both the terminal price S_T and the path $(S_t)_{t \leq T}$ determine the terminal payoff. If S_t follows a geometric Brownian motion, then \overline{S} is the infinite sum of infinitesimal lognormally distributed random variables, and the sum or average of log-normal random variables is very difficult to express analytically. There is no general closed form solution for the price of this option [4]. There are, however, analytic solutions where \overline{S} is the *geometric* average, that is, in discrete time

$$\overline{S} = (S_1 S_2 \dots S_T)^{1/n}$$

Here, $\overline{S} = \exp\{\frac{1}{T}\sum_{t=1}^{T}\log(S_t)\}\)$ and if S_t are lognormally distributed, we are summing normal random variables in the exponent. Thus the geometric average is lognormally distributed.

Our objective is to numerically price arithmetic-average Asian options. Unlike Monte Carlo pricing of European options where we could price by simply drawing from the S_T log-normal distribution, now we must simulate entire Brownian sample paths to obtain the distribution of \overline{S} . The computational burden of doing this is very large. Each Monte Carlo point estimate now comprises hundreds of sampling operations instead of just one.

6.4.1 Example: Pricing an Asian Option

Consider the option with the same parameters as before: $S_0 = 100, K = 100, r = 0.1, \sigma = 0.1$ and T = 1. Set N = 10000. We can price this option in R by

```
n<-round(N/2)
sim<-matrix(nrow=N,ncol=n)
Discounted<-1:N
for (i in 1:N)
    sim[i,]<-(sde.sim(t=T,n=n,x0=S,mu=function(X,t){x*r},sigma=function(x,t){x*sigma}))
for (i in 1:N)
    R[i]<-exp(-T*r)*max(z*(sim[i,n]-mean(sim[i,])),0)
mean(R)
[1] 2.20230</pre>
```

This algorithm is implemented in MCsim with option type "cas" for call and "pas" for put Asian options.

7 Conclusion

This paper serves as an introduction to Itô processes, numerical solution of stochastic differential equations, and option pring though simulation of Itô processes. Two numeric approximations of X_t , the Euler and the Milstein, are derived through Itô's lemma. It shows that the Euler has an error order $O(\Delta t)$ and the Milstein has error order $O(\Delta t)^{3/2}$. The Euler approximation is used in conjunction with Monte Carlo methods to estimate the price of Asian options. The price estimates obtained from Monte Carlo simulation are compared to the analytic Black-Scholes formula.

A Appendix

A.1 Solving the Geometric Brownian Motion SDE through Coefficient Matching

The coefficient matching method works only for linear μ and σ functions. Here's an example for geometric Brownian motion. Let $Y_t \equiv f(W_t, t)$. Applying Itô's lemma on $f(W_t, t)$ yields

$$dY_t = f(0,0) + \{\frac{1}{2}f_{xx}(x,t) + f_t(x,t)\}dt + f_x(x,t)dW_t$$

We want $dY_t = rY_t dt + \sigma Y_t dW_t$, so matching the coefficients of dt and dW_t looks tempting. Lets try it:

$$\begin{aligned} \sigma f(x,t) &= f_x(x,t) \Rightarrow f(x,t) = x_0 e^{\sigma x + g(t)} \\ rf(x,t) &= \frac{1}{2} f_{xx}(x,t) + f_t(x,t) \Rightarrow rf(x,t) = \frac{\sigma}{2} f_x(x,t) + f_t(x,t) \end{aligned}$$

Now $\frac{\sigma}{2}f_x(x,t) + f_t(x,t) = x_0 \frac{\sigma^2}{2} e^{\sigma x + g(t)} + g'(t)x_0 e^{\sigma x + g(t)}$, so we have

$$re^{\sigma x+g(t)} = \frac{\sigma^2}{2}e^{\sigma x+g(t)} + g'(t)e^{\sigma x+g(t)},$$

and we conclude that $g(t) = (r - \frac{\sigma^2}{2})t$. The solution is thus

$$f(W_t, t) = x_0 e^{(r - \sigma^2/2)t + \sigma W_t}$$

B Program Code

This section presents all the *R*-based functions written for the report. The main function is **sde.sim** which produces a sample solution to any SDE specified by $\mu(x, t)$ and $\sigma(x, t)$. **MCsim** performs Monte Carlo pricing of European and Asian call and put options using sde.sim to generate sample price paths.

B.1 sde.sim

Generates a sample solution to any SDE specified by functions $\mu(x, t)$ and $\sigma(x, t)$ over [0, t] with with n sample points.

 $\label{eq:sde.sim} \begin{aligned} & sde.sim <-function(t,n=1000,mu=function(x,t)\{0\},sigma=function(x,t)\{1\},x0=1,type="Euler",plot=T,innov=c()) \end{aligned}$

```
\begin{array}{l} dw <-innov \\ t <-seq(0,t,length=n) \\ dt <-(t[2]-t[1]) \\ x <-1:n \\ x[1] <-x0 \\ h <-10^{-6} \\ if (is.null(innov)) \\ dw <-rnorm(n,sd=sqrt(t/n)) \\ \end{array}
\begin{array}{l} if (type == "Euler") \\ for (i in 2:n) \\ x[i] <-(x[i-1]+mu(x[i-1],t[i])*dt+sigma(x[i-1],t[i])*dw[i]) \\ \end{array}
\begin{array}{l} if (type != "Euler") \\ \end{array}
```

B.2 Monte Carlo Simulation

B.2.1 BSEuro

Uses the Black-Scholes formula to calculate call and put European option value with specified initial price S, strike K, time to maturity T, annual interest rate r, volatility sigma. Type "pe" gives the put value, "ce" gives the call value.

```
BSeuro<-function(S,K,T,r,sigma,type=c("ce","pe")){
```

```
if (type!="ce" && type!="pe")
    stop("type misspecified: ce|pe")
d1<-(log(S/K)+(r+1/2*sigma^2)*(T))/(sigma*sqrt(T))
d2<-(d1-sigma*sqrt(T))
if (type=="ce")
    value<-(S*pnorm(d1,mean=0,sd=1)-K*exp(-r*(T))*pnorm(d2,mean=0,sd=1))
if (type=="pe")
    value<-(-S*pnorm(-d1,mean=0,sd=1)+K*exp(-r*(T))*pnorm(-d2,mean=0,sd=1))</pre>
```

result<-list(rbind("period"=T,"current price"=S,"strike"=K, "interest rate"=r,"volitility"=sigma,"type"=type),"Option Value"=value) return(result)}

B.2.2 MCsim

Performs Monte Carlo simulation to price European, Asian and fixed-strike average put and call options with with specified intial price S, strike K, time to maturity T, annual interest rate r, volatility sigma. If "boot"=TRUE, gives non-parametric bootstrap confidence intervals for the price estimate.

MCsim<-function(S,K,T,r,sigma,N,type=c("ce","cas","cav","pe","pas","pav"),boot="FALSE"){

z=NA

```
if (type == "ce" || type == "cas" || type == "cav")
   z = +1
if (type == "pe" || type == "pas" || type == "pav")
   z = -1
if (is.na(z))
   stop("type misspecified: ce|cav|cas|pe|pav|pas")
if (type == "ce" || type == "pe")
   sim < -sde.sim(n=N,t=T,mu=function(x,t){r*x},sigma=function(x,t){sigma*x}
   R < -exp(-T*r)*pmax(z*(sim-K),0)
else {
   n < round(N/3)
   sim<-matrix(nrow=N,ncol=n)
   for (i \text{ in } 1:N)
      sim[i,]<-(gen.brown(S,T,n,r,sigma,type="geo"))
   R<-1:N
   if (type == "cas" || type == "pas"){
      for (i \text{ in } 1:N)
      R[i] < \exp(-T^*r)^* \max(z^*(\min[i,n]-\max(\min[i,j])), 0)
      }
   if (type == "cav" || type == "pav")
      for (i \text{ in } 1:N)
      R[i] < -exp(-T*r)*max(z*(mean(sim[i,])-K),0)
      }
   }
plot(density(R))
```

```
if (type == "ce" || type == "pe"){
    BS<-BSeuro(S,K,T,r,sigma,type)$O
    summary<-rbind(
    "Estimate"=mean(R),
    "BS exact"=BS,
    "variance"=var(R),
    "bias"=(mean(R)-BS))}</pre>
```

else {

summary<-rbind(
"Estimate"=mean(R),</pre>

```
"variance"=var(R))}
if (boot=="TRUE"){
  sampmean<-function(p,x) {sum(p*x)/sum(p)}
  if (length(R)<=2000)
      conf<-abcnon<sup>3</sup>(R,sampmean)$limit
  else
      conf<-abcnon(R[1:2000],sampmean)$limit
  summary<-list(summary,"Bootstrap Conf Int."=conf)
  }
return(summary)</pre>
```

```
}
```

References

- [1] McLeish, D., "Monte Carlo Simulation and Finance", Wiley & Sons, 2005.
- [2] Boyle, P., "Options: A Monte Carlo Approach", Journal of Financial Economics, 4, 1977, 323-338.
- [3] Niederreiter, H., "Quasi-Monte Carlo methods and pseudo-random numbers", Bull. Am. Math. Soc. 84, 1978, 957-1041.
- [4] Kemma, A., Vorst, A., "A pricing method for options based on average asset values", Journal of Banking and Finance, 14, 1990, 113-129.

 $^{{}^{3}}$ **abcnon** part of the R *bootstrap* library. Generates non-parametric bootstrap confidence intervals for the sample mean.