Stat 251 Paper Review: "A Jump Diffusion Model for Option Pricing," S. G. Kou

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Abstract

This paper surveys the family of Jump Diffusion (JD) models for price variation. It focuses on the Double Exponential Jump Diffusion (DEJD) model proposed by Kou (2002). This simple model has several nice features allowing for analytic pricing solutions for many options, notably exotic and path-dependent. The features and specification of the DEJD model are discussed and placed into the general framework of Affine Jump Diffusion models proposed by Duffie (1995). The approximate returns distribution is derived and compared with the normal. It looks at equilibrium conditions imposed on jump diffusions under the rational expectations model. Simulations of the DEJD model are presented and compared to the S&P 500 index. Empirical testing and estimation of the model parameters is discussed.

1 Introduction

Almost every aspect of modern finance, from valuations and portfolio choice to option pricing and corporate finance, critically depend on the distribution describing the dynamics of security prices. For nearly a century, this distribution was normal. The Brownian motion model, first proposed in Bachlier's thesis *Theorie de la Speculation* (1900) and the geometric Brownian motion model (GBM) first proposed by Osborne (1959), have been enshrined in Finance's two most successful theories: the Capital Asset Pricing Model (Sharpe, 1964) and the Black-Scholes Model (Black and Scholes, 1972)

Despite the success of the Black-Scholes model, empirical evidence against GBM has accumulated; see, among others, Sundaresan, 2000. Specifically, (1) real returns are not normally distributed, instead the tails are asymmetric and heavier than normal (called "leptokurtotic") , (2) volatility is not constant, instead it clusters into high and low levels and displays long range dependence, (3) a "volatility smile" in calculating implied volatilies through the Black-Scholes framework exists, see Bakashi et al. (1997) for an overview.

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The inadequacy of Brownian motion has led to development of many alternative continuous-time models for price variation. These include stochastic volatility models, see Duffie (1995); conditional heteroskedastic models, in particular ARCH (Engle, 1982) and GARCH (Bollerslev, 1986); a chaos theory, multi-fractal model, see Mandelbrot (1963); and jump-diffusion models (JD), see Merton (1976a) and Kou (2002).

The jump-diffusion model was first suggested by Merton (1976a, 1976b), following the work of Press (1967). Merton suggested that returns processes consist of three components, a linear drift, a Brownian motion representing "normal" price variations, and a compound Poisson process accounting for "abnormal" changes in prices (jumps) generated by important information arrivals. Upon each information arrival, the jump magnitude is determined by sampling from an independent and identically distributed (i.i.d) random variable.

A large class of JD models have been proposed; see Duffie (1995) and Merton (1990) for reviews. Merton's first 1976 model assumes that this distribution is log-normally distributed (LND), which renders estimation and hypothesis testing tractable. It has also been shown to be consistent with empirical return distributions, displaying higher peaks, excess kurtosis and skewness; see Bakshi, Cao & Chen, 1997. However, analytic pricing formulas for only very simple options, such as European and American, exist under this distribution (Kou, 2002). Proposed variations on Merton's model include different distributional assumptions for jump magnitude (Ramezani & Zeng, 1998; Andersen & Andereasen, 2000; Kou, 2002), time-varying jump intensity, correlated jump-size (Naik (1993) and models where both price and volatility jump (Bakshi & Cao, 2004 and Eraker et al., 2003).

Each of these specific formulations are special cases of the Affine Jump Diffusion (AJD) framework proposed by Duffie, Pan and Singleton (2000). This framework is popular among researchers due to its modelling flexibility and its tractability in deriving risk-neutral representations for options pricing, as well as for straight-forward econometric estimation.

The Double Exponential Jump Diffusion (DEJD) model proposed by Kou (2002) is a special case of the AJD family. The model model enjoys several nice properties. The returns implied by the model are leptokurtotic and asymmetric. The model fits observed volatility smiles well. The model can be embedded into a rational expections equilibrium framework. Most importantly, the memoryless property of the exponential distribution family allows pricing formulas for exotic and path-dependent to be obtained exactly. This is a significant advantage over alternative JD models in which analytic tractability is confined to plain vanilla options. Because of these features, the DEJD is a popular choice within the JD family and has been used to model price variation in stock indices (Ramezani & Zeng, 2004) and bond yield spread (Huang & Huang, 2003).

The remainder of this paper considers the DEJD model in detail. It summarizes the main results of Kou (2002), the returns distribution under jumps, that DEJD can be embedded into a rational expectations equilibrium framework, and pricing formulas for European options. No empirical work was carried out to test the DEJD in Kou's study. I look at the work of Ramexani and Zeng (2004), which empirically assesses the DEJD model against LJD and ARCH alternative models. This is the only empirical study to date to consider the DEJD model. They conclude the results are, at best, weak.

The paper is structured as follows. Section 2 presents the Merton and the Double Exponential Jump Diffusion models. A simulation is carried out to graphically compare the returns distribution under DEJD to GBM and real returns from the S&P-500 index. Section 3 solves the DEJD stochastic differential equation and, from this solution, derives approximations to the returns density. It presents Kou's exact density result, and provides an illustrative example comparing the DEJD returns density to GBM returns density. Section 4 briefly looks at general equilibrium theory and the rational expectations model (Lucas, 1978), and considers the result the log of the jump distribution must belong in the exponential family. Section 5 discusses the problems of market incompleteness under jump-diffusions and existence of risk-free measures, and derives the discounted payoff expectation equal to the price of a European call option. Kou's formula for evaluating this expectation is given. Finally, section 6 surveys the empirical work done in assessing the DEJD model compared to other JD models and ARCH specifications.

2 The Jump-Diffusion Model

2.1 Merton's Model

The jump-diffusion model on which Kou's model is based was described by Merton (1976), which we follow here. The model for price variation consists of two parts: (1) a continuous diffusion according to geometric Brownian motion and (2) random time discontinuous jumps. The economic reasoning behind adding jumps is that these correspond to important information revelations that cause the market to quickly revalue the underlying asset. Under Merton's model, the jump process is specified by a Poisson process with intensity λ, thus

$$P(\text{jump in } [t+\Delta t)) = \lambda \Delta t + o(\Delta t).$$

and the times between jumps are i.i.d exponential random variables. Upon the i^{th} event time, the jump magnitude Y_i is drawn i.i.d from the log-normal distribution and the price changes from S_t to $S_t Y_i$. Merton notes that under the log-normal assumption, the distribution of S_t/S_o is lognormal with random variance parameter following the Poisson distribution.

2.2 Double Exponential Jump Diffusion Model

Kou proposes simple jump-diffusion model where the price of a financial asset is modelled by two parts, continuous geometric Brownian motion and jumps at random times with logarithm of jump sizes having double exponential distribution. The DEJD model has several nice features compared to alternative JD models. The returns distribution are asymmetric and leptokurtotic. Kou shows that the "volatility smile" exists under simulated prices with double exponential jumps. Most importantly, the model has good analytic tractability.

Specifically, under the DEJD model, the price of an underlying asset, S_t, is assumed to satisfy the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left\{\sum_{i=1}^{N(t)}(V_i - 1)\right\} v \tag{1}$$

where W_t is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ and $\{V_i\}$ is a sequence of independent and identically distributed nonnegative random jump sizes. The $\mu dt + \sigma dW_t$ specifies a continuous geometric Brownian motion with instantaneous μ drift and σ volatility. The random increment $d\sum_{i=1}^{N(t)}(V_i - 1)$ produces a jump $(V_i - 1)$ during time $t \in [s, s + ds]$ upon the event $N(s + ds) = 1 + N(s)$.

The distribution of $Y = \log(V)$ is asymmetric double exponential with density

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \;\; \eta_1 > 0, \;\; \eta_2 > 0 \tag{2}$$

where $p, q \geq 0$ and $p + q = 1$, represent the probability of upward and downward jumps. This is equivalently,

$$Y = \begin{cases} \xi^+, & \text{with probability } p \\ -\xi^-, & \text{with probability } q \end{cases},$$

where ξ^+ and ξ^- are exponential random variables with rates η_1 and η_2 respectively. This distribution was first proposed by Laplace (1774), giving rise to another name – "the first law of Laplace", whereas the "second law of Laplace" is the normal density. The processes W_t, $N(t)$, and V_i are assumed to be independent. Some useful properties of Y and V are

$$E(Y) = \frac{p}{\eta_1} - \frac{q}{\eta_2}, \;\; \mathrm{Var}(Y) = pq(\frac{1}{\eta_1} - \frac{1}{\eta_2})^2 + (\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}),$$

$$E(V) = E(e^Y) = q\frac{\eta_2}{\eta_1 + 1} + p\frac{\eta_1}{\eta_1 - 1}, \;\; \eta_1 > 1, \;\; \eta_2 > 0.$$

Example Consider figure (1), which shows three series of plots: the log prices and returns of the S&P 500 index from 1950 to 2006, log prices and returns generated by the DEJD model. The simulated returns from DEJD are generated by discrete approximation of the SDE, see Pollard (2006*a*) for details, and *R* code to do this is presented in A.2 of the Appendix.

The parameters for the DEJ with $\mu = 20\%$ per annum, $\sigma = 20\%$ per annum, $\Delta t = 1$ day, $\lambda = 20$, $\kappa = -0.02$ and $\eta = 0.02$. That is, I assume that there are about 20 daily jumps per year with average jump size -2% and with jump volatility 2%.

It is clear that GBM returns bear little resemblance to real returns – to recently quote Mandelbrot, "nothing happens" in GBM. The DEJD model seems to perform considerably better. The real returns experience occasional large jumps, such as October 1987's Black Tuesday, and the DEJD model captures. However, close inspection of real returns reveals that the volatility of "normal" variation oscillates between high and low slowly between years. The DEJD is not capturing this additional dynamic in prices

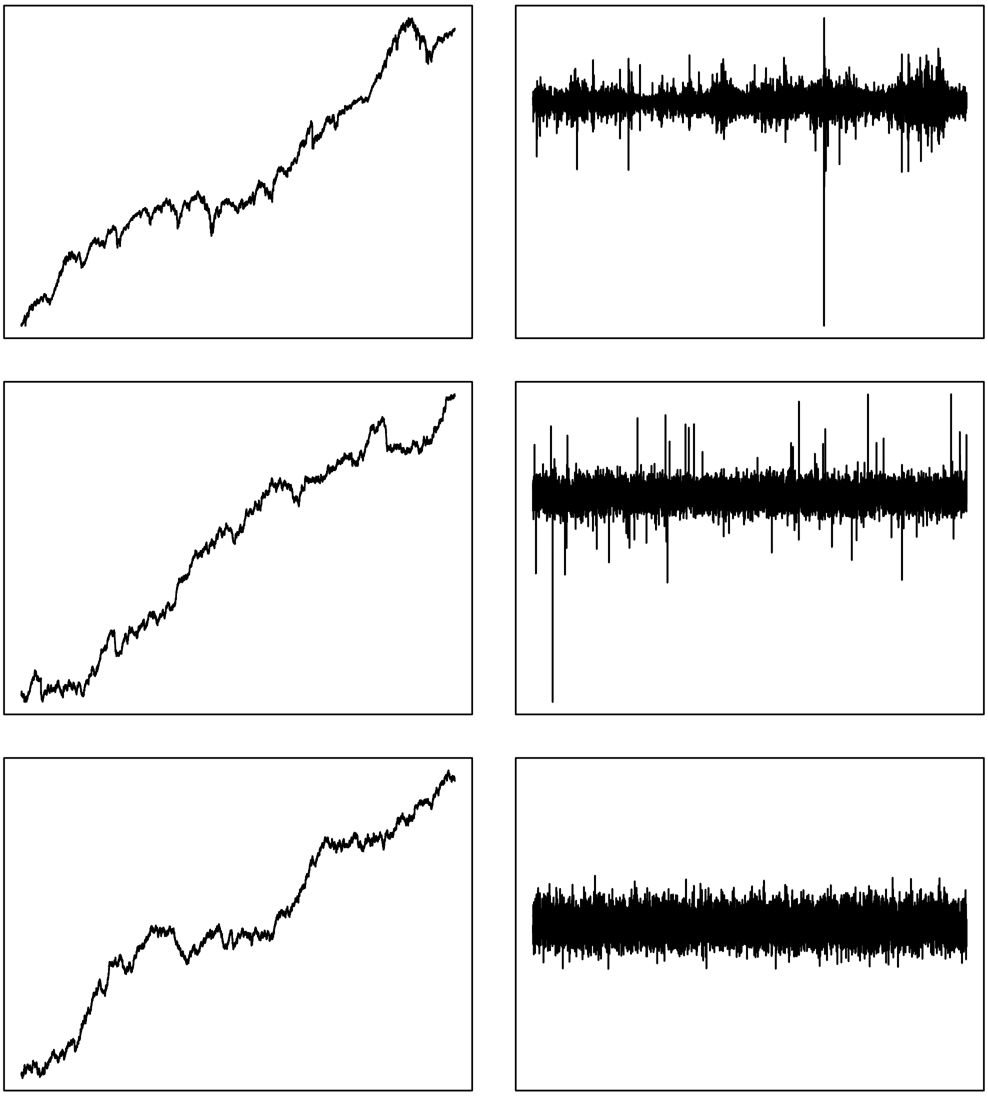

Figure 1: *Top:* S&P 500, log-prices left and returns right; *Middle:* Jump-Diffusion model, log-prices left and returns right; *Bottom:* geometric Brownian motion model, log-prices left and returns right.

variation.

3 Return Distribution

The solution to the stochastic differential equation in equation (1) is given by

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\} \prod_{i=1}^{N(t)} V_i. \tag{3}$$

This result generalizes the solution for GBM, $S_t = S_0\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$, to include stochastic jumps. Proof of (3) is given in A.1 of the Appendix. From (3) , the simple return of the underlying asset in a small time increment Δt becomes

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \exp\left[(\mu - \sigma^2/2)\Delta t + \sigma(W_{t+\Delta t} - W_t) + \sum_{i=N(t)+1}^{N(t+\Delta t)} Y_i\right] - 1$$

where $X_i = \log(V_i)$. For small Δt, we have the approximation $e^x \approx 1 + x + x^2/2$ and the result $(\Delta W_t)^2 \approx \Delta t$ to obtain

$$\begin{aligned} \frac{S_{t+\Delta t} - S_t}{S_t} &\approx (\mu - \sigma^2/2)\Delta t + \sigma \Delta W_t + \sum_{i=N(t)+1}^{N(t+\Delta t)} X_i + \frac{1}{2}\sigma^2(\Delta W_t)^2 \\ &\approx \mu\Delta t + \sigma Z\sqrt{\Delta t} + \sum_{i=N(t)+1}^{N(t+\Delta t)} X_i, \end{aligned}$$

where Z is a random variable with standard normal distribution and $\Delta W_t = W_{t+\Delta t} - W_t$.

Under the assumption $N(t)$ follows a Poisson process, the probability of having one jump in the interval $(t, t + \Delta t]$ is $\lambda\Delta t$ and having more than one is $o(\Delta t)$. Therefore, for small Δt, we have

$$\sum_{i=N(t)+1}^{N(t+\Delta t)} X_i \approx \begin{cases} X_{N(t)+1} & \text{with probability } \lambda\Delta t \\ 0 & \text{with probability } 1 - \lambda\Delta t \end{cases}$$

and combining these results, the simple return of the underlying asset is approximately distributed as

$$\frac{S_{t+\Delta t} - S_t}{S_t} \approx \mu\Delta t + \sigma Z\sqrt{\Delta t} + I \times X, \tag{4}$$

where I is a Bernoulli random variable with $P(I = 1) = \lambda\Delta t$. Equation (4) allows for simple simulation of the returns generated by the DEJD model. Simulations are presented in figure (1), discussed in Section 7. The function used to simulate returns and code is presented in A.2 of the Appendix.

Let G equal the right hand side of (4). Using the independence between the exponential and normal distributions used in the model and formulae for the sum of double exponential random variables, Kou (2002) obtains the probability density function of G

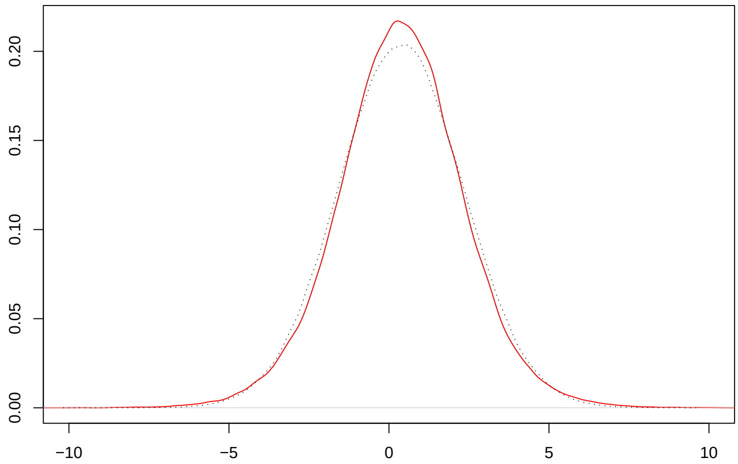

Figure 2: Density $g(x)$ for a double exponential (*red*) with $\mu = 0.1$, $\lambda = 1$,$\Delta t = 1$ $\kappa = 0.1$, and density of normal (*black, broken*) with equal mean and variance.

as

$$\begin{aligned} g(x) \quad &= \quad \frac{\lambda \Delta t}{2\eta} e^{\sigma^2 \Delta t/(2\eta^2)} \left\{ e^{-\omega/\eta} \Phi\left(\frac{\omega\eta - \mu\Delta t}{\sigma\eta\sqrt{\Delta t}}\right) + e^{\omega/\eta} \Phi\left(\frac{\omega\eta + \mu\Delta t}{\sigma\eta\sqrt{\Delta t}}\right) \right\} \\ & \quad +(1-\lambda\Delta t)\frac{1}{\sigma\sqrt{\Delta t}} f\left(\frac{x-\mu\Delta t}{\sigma\sqrt{\Delta t}}\right), \end{aligned}$$

where $\omega = x - \mu\Delta t - \kappa$, and $f(.)$ and $\Phi(.)$ are the probability density and cumulative distribution functions of the standard normal random variable. The density g has mean and variance given by

$$E(G) = \mu\Delta t + \lambda\Delta t\kappa$$

$$Var(G) = \sigma^2\Delta t + 2\eta^2\lambda\Delta t + \kappa^2\lambda\Delta t(1-\lambda\Delta t).$$

The density $g(x)$, comparing to a normal density with equal mean and variance, has a higher peak and two heavy tails. The distribution is not symmetric: if the mean jump size $\kappa > 0$, the distribution is skewed right; and if $\kappa < 0$, skewed left. An example of $g(x)$ with parameters $\mu = 0.1$, $\lambda = 1$,$\Delta t = 1$ $\kappa = 0.1$ is drawn in figure (2) with a normal density overlayed.

4 Equilibrium under Jump Diffusion

The prices of any asset, financial or otherwise, is subject to forces of demand and supply. By guidance of the market's "invisible hand", the adjustment of prices brings these forces to equality. The point where demand and supply match, the equilibrium point, needs to exist, be unique and determinable if asset prices, such as options prices, are to be predictable from theory.

Economists classify market equilibrium models into two categories, partial and general Partial equilibrium models take the variables of prices, expectations and preferences as given, and derive results that are assumed not to affect these market variables. Option pricing models are typically partial equilibrium models, and a famous example is the Black-Scholes model. Here, the expected return of a security, the risk free interest rate and the volatility are all determined independently of the model, and investor preferences do not matter.

General Equilibrium models provide results by aggregating across assets and market participants. They take the existence of a class of assets, the prices of those assets, and the expectations and preferences of all market participants as variables that are determined by the model itself. The simplest general equilibrium model is called the Rational Expectations model (Lucas, 1978) and forms the basis of the Efficient Market Hypothesis.

Kou considers the family of jump diffusion models and whether a market equilibrium exists under rational expectations model. Under this model, there is only one risky asset with price $p(t)$. People ("agents") are homogeneous and each try to solve the utility maximization problem

$$\max_c E\left[\int_0^\infty U(c(t),t)dt\right]$$

where $U(c(t),t)$ is the utility function of the consumption process $c(t)$. Each agent has an exogenous endowment process, $\delta(t)$, which provides income that can be either consumed or invested in a single risky asset that pays no dividends. If δ is Markovian, Stokey and Lukas (1989) show that under mild conditions, $p(t)$ must satisfy the Euler equation

$$p(t) = \frac{E[U_c(\delta(T),T)p(T)|\mathcal{F}_t]}{U_c[\delta(t),t]}, \quad \forall T \geq t \geq 0, \tag{5}$$

where U_c is the marginal utility of consumption at time t.

Kou explicitly derives the implication of equation (5) when the endowment process $\delta(t)$ and the price S_t of the underlying asset are stochastic and satisfies the jump-diffusion SDE in (1) with jumps $(\widetilde{V}_i - 1)$ with $\widetilde{V}_i$ having arbitrary distribution. Specifically, he proposes a special correlated jump diffusion form for $S(t)$,

$$\frac{dS_t}{S_t} = \mu dt + \sigma(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\} + d\left\{\sum_{i=1}^{N(t)}(V_i - 1)\right\}, \quad V_i = \widetilde{V}_i^{\beta} \tag{6}$$

where the power $\beta \in (-\infty, \infty)$ is an arbitrary constant, where W_t^1 is independent of W_t^2 and W_t^1 drives the jump diffusion for $\delta(t)$ given by (1). He considers equilibrium only under special utility function forms

$$U(c,t) = \frac{e^{-\theta t}c^\alpha}{\alpha}, \quad 0 < \alpha < 1, \quad U(c,t) = e^{-\theta t}\log(c), \quad \text{if } \alpha = 0.$$

Kou's first theorem states that the model in (6) satisfies the equilibrium conditions if and only if

$$\mu = r + \sigma_1\sigma_2\rho(1-\alpha) - \lambda(\varphi_1^{(\alpha+\beta-1)} - \varphi_1^{(\alpha-1)})$$

where $\varphi_1^{(a)} := E[(\widetilde{V}_i^a - 1)]$, the expected proportional jump size affecting the endowment $\delta(t)$.

The second important result specifies possible distributions the endowment jump, $\widetilde{V}$. Let $\mathcal{V}$ be the family distributions for $\widetilde{V}$. Then if for any real number a,

$$\widetilde{V}^a \in \mathcal{V}, \;\; kx^a f_{\widetilde{V}}(x) \in \mathcal{V}$$

where $k = \{\varphi_1^{(a-1)} + 1\}^{-1}$,then the jump sizes for the asset price S_t under the physical probability measure P and jump sizes for S_t under the rational expectation risk-neutral measure Q belong to the same family $\mathcal{V}$. Kou notes that this essentially requires $Y = \log(V)$ to belong to the exponential distribution family. This condition satisfied both with Y having normal distribution, as in the LJD model, and Y having double exponential distribution.

5 Option Pricing

A serious criticism of JD models is that under the presence of random jumps the market becomes incomplete. This mean that replication of an option, or equivalently, riskless hedging of an option position, is impossible since prices discontinuously jump. However, Kou argues that this should be considered only as a special property of the Brownian Motion framework, and since riskless hedging is impossible in discrete time, this loss should not make a difference to the actual value of an option. This is also voiced by Merton (1976a) and Naik and Lee (1990).

Given incompleteness, we can still derive option pricing formula that does not depend on risk attitudes of investors. Merton (1976a) proposes an economic argument: that, if the number of securities available is very large, the risk of sudden jumps is diversifiable and the market will therefore pay no risk premium over the risk-free rate for bearing this risk. Alternatively, for a given set of risk premiums, we can consider a risk-neutral measure Q so that under Q

$$\frac{dS_t}{S_t} = [r - \lambda\varphi]dt + \sigma dW_t + d\left\{\sum_{i=1}^{N(t)}(V_i - 1)\right\}$$

where r is the risk-free interest rate, σ is price volatility between jumps, λ is jump intensity,

$$\varphi = E[V-1] = \exp(\kappa)/(1-\eta^2) - 1$$

where $\kappa = E(Y)$, and the parameters r, σ, λ, κ and η become risk-neutral parameters taking consideration of the risk premiums. Kou constructs Q under the rational expectations framework as follows. Let $\delta(t)$ be a stochastic endowment process satisfying the DEJD SDE as before, and define

$$Z(t) := e^{rt}U_c(\delta(t), t) = e^{(r-\theta)t}(\delta(t))^{\alpha-1}$$

where θ and α are parameters in investor's intertemporal consumption utility function.

Then $Z(t)$ is a martingale under P, and define the measure Q by the Radon-Nikodym derivative

$$\frac{dQ}{dP} := \frac{Z(t)}{Z(0)}.$$

Then under Q, the Euler equation (5) holds if and only if

$$S_t = e^{-r(T-t)} E_Q(S_T|\mathcal{F}_t).$$

The unique solution of this SDE is

$$S_t = S_0 \exp[(r - \sigma^2/2 - \lambda\varphi)t + \sigma dW_t] \prod_{i=1}^{N(t)} V_i,$$

I prove this in A.1 of the Appendix.

To price a European option in DEJD model, it remains to compute the expectation, under measure Q, the discounted expectation of final payoff of the option. For a European call, the price at time t equals

$$\begin{aligned} P_t^c &= E_Q[e^{-r(T-t)}(S_T - K)_+] \\ &= E_Q\left[e^{-r(T-t)}\left(S_t \exp\left\{\frac{r-\sigma^2}{2-\lambda\varphi}(T-t) + \sigma\sqrt{T-t}Z\right\}\prod_{i=1}^{N(t)} V_i - K\right)_+\right], \end{aligned}$$

where T is the expiration time, $(T-t)$ is the time to expiration measured in years, K is the strike price, $(y)_+ = \max(y, 0)$ and Z is a standard normal random variable. Kou shows that P_t is analytically tractable as

$$\begin{aligned} P_t^c &= \sum_{n=1}^{\infty}\sum_{j=1}^{n} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \frac{2^j}{2^{2n-1}} \begin{pmatrix} 2n-j-1 \\ n-1 \end{pmatrix} \times (A_{1,n,j} + A_{2,n,j} + A_{3,n,j}) \\ &\quad + e^{-\lambda(T-t)}\left[P_t e^{-\lambda\varphi(T-t)}\Phi(h_+) - Ke^{-r(T-t)}\Phi(h_+)\right], \end{aligned}$$

where $\Phi(.)$ is the standard normal cumulative distribution function,

$$A_{1,n,j} = P_t^c e^{-\lambda\varphi(T-t)+n\kappa}\frac{1}{2}\left(\frac{1}{(1-\eta)^j} + \frac{1}{(1+\eta)^j}\right)\Phi(b_+) - e^{-r(T-t)}K\Phi(b_-)$$

$$A_{2,n,j} = \frac{1}{2}e^{-r(T-t)-\omega/\eta+\sigma^2(T-t)/(2\eta^2)K} \times \sum_{i=0}^{j-1}\left\{\frac{1}{(1-\eta)^{j-i}} - 1\right\}\left(\frac{\sigma\sqrt{T-t}}{\eta}\right)^i \frac{1}{\sqrt{2\pi}} Hh_i(c_-)$$

$$A_{3,n,j} = \frac{1}{2}e^{-r(T-t)+\omega/\eta+\sigma^2(T-t)/(2\eta^2)K} \times \sum_{i=0}^{j-1}\left\{1 - \frac{1}{(1-\eta)^{j-i}}\right\}\left(\frac{\sigma\sqrt{T-t}}{\eta}\right)^i \frac{1}{\sqrt{2\pi}} Hh_i(c_+)$$

$$b_\pm = \frac{\ln(S_t/K) + (r \pm \sigma^2/2 - \lambda\varphi)(T-t) + n\kappa}{\sigma\sqrt{T-t}}$$

$$
\begin{aligned}
h_{\pm} &= \frac{\ln(S_t/K) + (r \pm \sigma^2/2 - \lambda\varphi)(T-t)}{\sigma\sqrt{T-t}} \\
c_{\pm} &= \frac{\sigma\sqrt{T-t}}{\eta} \pm \frac{\omega}{\sigma\sqrt{T-t}} \\
\omega &= \ln(K/P_t) + \lambda\varphi(T-t) - (r-\sigma^2/2)(T-t) - n\kappa \\
\varphi &= \frac{e^{\kappa}}{1-\eta^2} - 1
\end{aligned}
$$

and the $Hn_i(.)$ functions are defined by

$$Hn_n(x) = \frac{1}{n!}\int_x^{\infty}(s-x)^n e^{-s^2/2}ds, \;\; n = 0, 1, ...$$

and $Hh_{-1}(x) = \exp(-x^2/2)$, which is $2\pi f(x)$ where $f(x)$ is the probability density function of the standard normal variable. The $Hh_n(x)$ functions satisfy the recursion

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x), \;\; n \geq 1,$$

The pricing formula involves an infinite series, but its numerical value can be approximated quickly and accurately through truncating the sum index n. Kou finds that $n = 10$ or 11 is usually sufficient to have price accuracy within a cent.

The formula is a generalization of the Black-Scholes formula. Specifically, there are two special cases that give the Black-Scholes formula exactly: (1) if $\lambda = 0$,that is, there are no jumps, and (2) $\kappa = 0$, $\eta \to 0$, that is , the jump sizes tend to zero. In the first case, we have

$$P_t^c = S_t\Phi(b_+^{'}) - Ke^{-r(T-t)}\Phi(b_-^{'}) \tag{7}$$

which is the Black-Scholes formula, where

$$b_{\pm} = \frac{\ln(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

In the second case, as $\eta \to 0$, $\varphi = \frac{1}{1-\eta^2} - 1 \to 0$. An application of the dominated convergence theorem and the fact that

$$\sum_{n=1}^{\infty} n\eta e^{-\lambda(T-t)}\frac{\lambda^n(T-t)^n}{n!} = \lambda T\eta \to 0,$$

shows that the P_t^c converges to the equation (7); see Kou (2002) appendix for details.

The price of a European put option under DEJD can be obtained by using the put-call parity

$$P_t^p = P_t^c + Ke^{-r(T-t)} - S_t.$$

Kou (2002) presents pricing formulas for other options. Specifically, he produces formulas for perpetual American options, barrier and loop-back options, and bond options.

6 Empirical Assessment

In his concluding comments, Kou (2002) notes that no empirical work was carried out to test the DEJD in his study. At the time of writing, only one study, Ramexani and Zeng (2004) ,has investigated the model against real data.

Ramezani and Zeng (2004) compare DEJD to six popular versions of the Autoregressive Conditional Heteroskedasticity (ARCH) and the Log-Normal Jump Diffusion (LJD) specification. They fit each model to daily data from a large sample of firms and monthly data from the S&P-500 and NASDAQ composite. The fit is carried out by maximum likelihood estimation and they utilize the BIC criterion to assess the performance of DEJD relative to the alternatives.

For individual stocks based on data spanning the period 10/96 to 12/98, they find both LJD and DEJD fit the data better than GBM for every firm, supporting the use of jump-diffusion models. Surprisingly, they find that that relative to LJD, DEJD provides a better fit for only 11% of the sampled firms. The results fell short of their expectations, given the flexibility associated with DEJD.

For comparison with ARCH specifications, they found that both DEJD and LJD perform better than the ARCH alternatives for the majority of individual stocks. For stock indexes, the ARCH specifications dominate, in particular, GARCH(1,1) dominate for monthly indexes and EGARCH(1,1) for daily indexes. The GBM specification fails to beat the alternatives for every time series.

In conclusion, they find that at best, the empirical evidence for DEJD is mixed.

7 Conclusion

This paper reviews the class of jump diffusion models paying particular focus on the Double Exponential Jump Diffusion process proposed by Kou (2002). Under this model, the price of a financial asset is modelled by two parts, continuous geometric Brownian motion and jumps at random times with logarithm of jump sizes having double exponential distribution. The DEJD model has several nice features compared to alternative JD models. The returns distribution are asymmetric and leptokurtotic. Kou shows that the "volatility smile" exists under simulated prices with double exponential jumps. Most importantly, the model has good analytic tractability, allowing for explicit calculation of both vanilla and path-dependent prices. The explicit calculation is made possible partly because of the memoryless property of the double exponential distribution. Furthermore, the model is compatible with a rational expectations framework unlike models using jump distribution outside the exponential family.

However, the work by Ramezani and Zeng (2004) suggest that the DEJD model is outperformed by Merton's log-normal jump diffusion specification in modelling the returns from individual stock and is dominated by GARCH(1,1) specifications for the S&P-500 and NASDAQ index. This is surprising given the flexibility of the DEJD model.

There are a number of interesting directions for future extensions of the model. Ramezani and Zeng (2004) suggest alternatives using time-varying jump intensities.

Adding correlation to the jump process may be used to simulate volatility clustering effects into the model. Alternatively, the plots in figure (1) suggest that a hybrid jump and stochastic volatility model are the best approach for empirical success. Of course, it should be noted that the main reason the DEJD model is attractive is its simplicity. Its unclear whether adding further bells and whistles improves things.

A Appendix

A.1 Solving the DEJD Model

The solution the stochastic differential equation in (**??**) is given by

$$S_t = S_0 \exp[(\mu - \sigma^2/2 - \lambda\varphi)t + \sigma W_t] \prod_{i=1}^{N(t)} V_i,$$

where $\prod_{i=1}^{0} = 0$. The solution is obtained as follows.

Let t_i be time corresponding to the i^{th} jump. For $t \in [0, t_1)$, there is no jump and the price is given by the solution to $dS_t/S_t = (r - \lambda\varphi)dt + \sigma dW_t$. Consequently, the left hand price limit at t_1 is

$$S_{t_1^-} = S_0 \exp(\mu - \sigma^2/2 - \lambda\varphi)t_1 + \sigma W_{t_1}].$$

At time t_1, the proportion of price jump is $V_1 - 1$ so the price becomes

$$S_{t_1} = (1 + V_1 - 1)P_{t_1^-} = V_1 P_{t_1^-} = P_0 \exp[(\mu - \sigma^2/2 - \lambda\varphi)t_1 + \sigma W_{t_1}]V_1.$$

For any $t \in (t_1, t_2)$, there is no jump in the interval $(t_1, t]$ so that

$$S_t = S_{t_1} \exp[(\mu - \sigma^2/2 - \lambda\varphi)(t - t_1) + \sigma(W_t - W_{t_1})].$$

Plugging in S_{t_1} yields

$$S_t = S_0 \exp([\mu - \sigma^2/2 - \lambda\varphi)t + \sigma W_t]J_1,$$

and repeating this scheme, we obtain the solution.

A.2 Program Code

A.2.1 rdexp

Draws random variates from the double exponential distribution specified by (2). The call variables are n (number of variates), p (skewness parameter, $p > 0$ means skewed right), and $\eta_1$and η_2 (rates of left and right exponential distributions).

```
rdexp = function(n,p,mu1,mu2) {
nleft = rbinom(1,n,p)
leftexp = -rexp(nleft,mu1)
```

```
rightexp = rexp(n-nleft,mu2)
s = sample(c(leftexp,rightexp),size=n,replace=F)
return(s) }
```

A.2.2 jumpdiff.sim

Simulates a price path according to the DEJD specification. Call variables: t is time, n isnumber of points to simulate mu is a function specifying $\mu(S_t, t)$ – in DEJD, $\mu(S_t, t) = rS_t$, default is $\mu = 0$. $sigma$ is a function specifying $\sigma(S_t, t)$. In DEJD, $\sigma(S_t, t) = S_t\sigma$, default is $\sigma = 1$. $x0$ is initial price, p is the skew parameter (as in rdexp), lambda is the intensity for the Poisson process. Mu1 and Mu2 are η_1,η_2 respectively.

```
jumpdiff.sim = function(t=50,n=1000,mu=function(x,t){0},sigma=function(x,t){1},
x0=1,plot=T,innov=c(),p=0.5,lambda=1,mu1=1,mu2=1){

dw<-innov
if (is.null(innov))
   dw<-rnorm(n,sd=sqrt(t/n))

nt<-sample(c(0,1),size=n,prob=c(1-t*lambda/n,t*lambda/n),replace=T)
tseq<-seq(0,t,length=n)
dt<-(tseq[2]-tseq[1])
doubleexp<-rdexp(n,p,mu1,mu2)
jumps<-(exp(nt*doubleexp)-1)

for (i in 2:n)
   x[i]<-(x[i-1]+mu(x[i-1],tseq[i])*dt+sigma(x[i-1],tseq[i])*dw[i]+(jumps[i]))

if (plot==T)
plot(cbind(tseq,x),type="l")
result<-list()
result$nt<-nt
result$x<-ts(x)
return(result)}
```

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