A Theory of Life by the Merton Model

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1 Model

- Consists of N consumtion "activities" and M jobs. Each activity yields μ_C expected utility and V_{CC} variance covariance *per dollar*, each job yields μ_W and V_{WW} expected return and variance-covariance.
- Allocation of wealth for consumption: $W_t x_t$
- Allocation of time between consumption and working at moment t :

$$\alpha Consumption + (1-\alpha) Work.$$

- Utility of Consumption:

$$dU_t = \alpha W_t x^T \mu_C dt + \alpha^2 W_t^2 x^T V_{CC} x dZ_t$$

- Income from working:

$$\frac{dI_t}{I_t} = (1-\alpha)\mu_W dt + (1-\alpha)^2 \mathbf{1}^T V_{WW} \mathbf{1} dZ_t$$

- Both working and consuming reduce health.
- Health H_t, depending on aggregate riskiness. Idea: all activities risky (reduce health), some faster than others (higher variance).
- Stochastic lifetime $T = \min(\tau : \ H_\tau = 0)$. Endogenous distribution (determined by optimal behaviour).
- Health dynamics:

$$dH_t = -Var(Life_t)dt + \nu dW_t.$$

  where $Var(Life_t) = \alpha_t^2 Var(C_t) + (1-\alpha_t)^2 Var(Work_t)$. Implies T has finite expectation. Distribution is complicated. ν exogenous.
- Maximize expected lifetime utility: $\max_x E \int_0^T U(C_t)dt$.
- Wealth dynamics: $dW_t = \underbrace{W_t dr_t}_{savings} + (I_t - C_t)dt$. May save at rate r_f.

Optimal allocation of consumption:

$$C_t \equiv W_t x_t = \underbrace{AV_{aa}^{-1}(\mu - r_f)}_{mean-variance\ allocation} - \underbrace{V_{aa}^{-1}V_{aH}H}_{health\ hedge} - \underbrace{V_{aa}^{-1}V_{aW}W}_{work\ hedge}$$

Optimal career path:

$$Work_t = \mathbf{max}(...)$$

Optimal work/play allocation

$$\alpha = \frac{U'(Work_t)}{U'(Work_t) - U'(C_t)}$$

Questions:

- What is the distribution of wealth and societal utility? Prob. exponential.
- What is the distribution of lifespans, how does it depend on risk aversion, initial wealth, job opportunities?
- If we price the activities based on demand with a fixed supply, what are the market prices. What is the clearing return/risk trade-off. Multi-CAPM representation for prices?

Extensions:

- Technology, state-variable X_t. This follows GBM as well. Depends on aggregate Hard-Work: $\mu_X(t) = f(\sum_{i=1}^{N_{people}} D_t^{(i)})$, e.g. linear function. Higher technology increases opportunity set of activities, i.e. N and M grow => allow people better risk-reward trade-offs.
  - Will induce extra savings (without this, optimal never to save due to risk of death).
  - Extra hedge term in consumption allocation, aka Merton model.

Model Behaviour:

The health process is locally predictable. Optimal behaviour near $H_t = 0$ is to do nothing and get 50% chance that H_t will rise. Reasonable? Maybe. Can model retirement: D_t also affects health negatively, when H_t near zero, optimal to neither work or do stuff.