Modelling Auctions with Independent and Private Values*

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29th November 2004

Abstract

This paper serves as an introduction to modern auction theory. The independent and private value model (IVP) is presented and used to derive equilibrium bidding strategy and seller's expected revenue in first-price, second-price, all-pay and war-of-attrition auctions. It proves that the expected revenue in each auction type is the same and presents the famous revenue equivalence theorem. The effects of introducing reserve prices on first-price and second-price auctions are analysed.

1 Introduction

This paper constructs a simple game-theoretic model of bidder behaviour and solves the four most important sealed bid auction formats: first-price, second-price, all-pay and war-of-attrition. Bidding strategy and expected seller's revenue is characterised for each from first principles and it is shown that each auction has equivalent revenue. Reserve pricing is analysed in the case of first- and second-price auctions and the optimal reserve price is found.

The paper is structured as follows. In section two, an auction is formally defined and types of auctions expounded. In section three, the Bayes-Nash equilibrium concept is introduced and used to derive equilibrium bidding strategy in section four. The equivalence of expected seller's revenue is shown in each auction and the Revenue Equivalence Theorem is discussed in section 4.2. Introducing reserve prices is formally analysed and revenue equivalence is shown to hold in section 4.3.

2 Definitions

A simple definition of an auction is an allocation system using prices determined by competition of buyers for the right to purchase. More formally it is defined

*The author thanks Sophie Turner and Michael Attock for their stimulating feedback. I am grateful for my supervisor Flavio Menezes for his support and patience.

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by (i) a pair of outcome functions, which allocate the good(s) and the bidder's payments, and (ii) a description of bidder's strategic plans (Myerson, 1981).

Under these broad definitions, there is a plethora of auction formats. First of all, whether an auction is open and sealed-bid can be distinguished. Under an *open* auction, bids are submitted openly, i.e. they are known by all participants. This is in contrast with a *sealed-bid* auction where participants only know their own private bids. The most common types of open auctioning is the English auction and the Dutch auction.

An *English* auction starts bidding at a low price and subsequent bids are of increasing amounts. The winner is the last bidder remaining. A *Dutch* auction does the reverse: an initial high price is gradually decreased until the winner makes the first (and only) bid.

The best known types of sealed bid auctions are the discriminatory and uniform-price auctions. In the *discriminatory* auction, N objects are allocated to bidders with the highest bids who pay the amount of their bid. Thus, for more than one object on sale, winning bidders pay varying prices in a discriminatory sense. If a single object is being sold, a discriminatory auction is also called *first-price* auction. The *uniform* auction differs from discriminatory auctions only in payment rule; here the N objects are priced identically at the highest unsuccessful bid. Thus all successful bidders pay the same amount, regardless of the amount they bid. A single object uniform auction is commonly called a *second-price* or *Vickery* auction. An important feature of the uniform auction is that bids determine only whether a bidder wins and not the actual payment.

Auction formats in which everyone makes a payment are special cases of discriminatory and uniform auctions. The *all-pay* auction is a discriminatory, single object auction where each bidder pays his or her bid. The *war-of-attrition* auction is a uniform, single object auction where the highest bidder pays the second highest (unsuccessful) bid and all other bidders pay their own bid.

Lastly, auctions can be classified as having private or common values. An object has *private values* if its value to a bidder is a purely personal matter, for example a painting. An object has a *common value* if all bidders share the same value for the object, however this value will generally be unknown at the time of the auction. Thus each bidder may make slightly different estimates and the common value can be treated as a random variable. A classic example is of companies competing for exploration rights to an oil field - each stands to benefit by the same monetary value if they win the right, however their individual estimates of this value will differ depending on what private information is available.

3 A game-theoretic model

The auctioneer or *seller* sets the auction rules and makes them clear to all participants. The auction rules are binding even though it may be in the seller's interests to change them after the bids have been submitted. The seller is risk

neutral - he or she is only concerned with maximising expected revenue from the sale. There are n, $n \geq 1$ bidders competing with each other in the auction. Each bidder has some private information, $v_i \in \mathbb{R}_+$, and information is considered a realisation of the random variable V_i. Bidders are distinguished solely by their private information; otherwise they are identical. Bidders do not cooperate and are risk-neutral. All bidders know the joint distribution of $(V_1, ..., V_n)$.

Bidders behave rationally to maximise personal profit, π. They take actions of competitors into account and behave strategically. The auction is considered to be a *non-cooperative game with incomplete information*. To analyse this game the Bayes-Nash equilibrium concept (Harsanyi, 1967) is employed. For that purpose, a *bidder's strategy* is defined as a positive real-valued function mapping his information, specifically his valuation, into bids. A *Bayes-Nash equilibrium* is defined as a n-tuple $(b_1, ..., b_n)$ of bidders' strategies with the property

$b_i : i = 1, ..., n$, maximises bidder i's expected profits assuming bidding strategies $b_1, ..., b_{i-1}, b_{i+1}, ..., b_n$of his competitors.

A *symmetric* Bayes-Nash equilibrium is such that all bidders choose the same bidding strategy (Menezes, 2004).

4 The auctions

4.1 The independent private value model

The private value of a good to a bidder i is denoted v_i and is a realisation of the variable V. Each bidder draws their v independently from the distribution $F_V(v)$ with density $f_V(v)$ and does not know any other bidder's valuation. If $P_i$denotes the payment bidder i pays if successful, and $I_{\{i\ successful\}}$denotes the indicator variable that equals 1 if the bidder wins the auction and zero otherwise, then the expected profits $\pi_i$are given by

$$E[(v_i - P_i)I_{\{i\ successful\}}] \tag{1}$$

Thus bidder i's expected profits can be written in the form

$$v_i \times Pr(winning) - expected\,payments \tag{2}$$

In a first price auction, the payment rule for the successful bidder is to pay his winning bid b. In a second price auction, the winner pays the second highest (unsuccessful) bid. In an all-pay auction, each bidder pays their bid regardless to whether they win. In a war-of-attrition auction, each bidder pays their bid except from the winner who pays the second highest bid.

4.1.1 First price auction

Analysing the auction is approached by considering the best response of bidder (bidder one) given all other bidders follow the strategy $\hat{b}(\cdot)$ which is strictly

increasing and continuous. Bidder one's best response is to bid in a manner that maximises his or her expected profits, π_1 given the object private value v_1. Bidder one's payoffs are

$$\pi_1 = \begin{cases} v_1 - b_1 & \text{if } b_1 > \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \\ 0 & \text{if } b_1 \leq \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \end{cases}$$

Assume that in the case of a draw, no one wins the auction. By (2) the expected profit is

$$\begin{aligned} \pi_1(b_1, v_1, b(\cdot)) &= (v_1 - b_1)Pr(b_1 > max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}) \\ &= (v_1 - b_1)Pr(b_1 > \hat{b}(v_2), ..., b_1 > \hat{b}(v_n)\}) \\ & \quad \text{by independence of } v_2, ..., v_n \end{aligned}$$

We know that $v_1 \in [0, \overline{v}]$ where $\overline{v} = max\{v_1, ..., v_n\}$, $\underline{v} = min\{v_1, ..., v_n\}$ therefore $b([0, \overline{v}]) = [\underline{b}, \overline{b}]$. This means there exists $x \in [0, \overline{v}]$ such that $\hat{b}(x) = b_1$. Bidder one's problem is equivalent to choosing $x \in [0, \overline{v}]$ to maximise $\pi(x)$.

$$\begin{aligned} \pi(x) &= (v - \hat{b}(x))Pr(\hat{b}(x) > \hat{b}(v_2), ..., \hat{b}(x) > \hat{b}(v_n)) \\ &= (v - \hat{b}(x))Pr(x > v_2, ..., x > v_n) \\ & \quad \text{since } \hat{b}(\cdot) \text{is a strictly increasing function} \end{aligned}$$

Again using independence of $v_2, ..., v_n$, the string of inequalities can be split up

$$\begin{aligned} \pi(x) &= (v - \hat{b}(x))\textstyle\prod_{i=2}^{n} Pr(x > v_i) = (v - \hat{b}(x))F_V(x)^{n-1} \\ &= (v - \hat{b}(x))F_V(x)^{n-1} \end{aligned}$$

Maximising $\pi(x)$ with respect to x gives

$$\frac{\partial \pi(x)}{\partial x} = (n-1)(v - \hat{b}(x))f(x)F(x)^{n-2} - \hat{b}'(x)F(x)^{n-1}$$

In symmetric equilibrium π is maximised at $x = v$, hence the first order condition is $\pi(v)' = 0$

$$\Rightarrow 0 = (n-1)(v - \hat{b}(v))f(v)F(v)^{n-2} - \hat{b}'(v)F(v)^{n-1}$$

$$\Rightarrow \hat{b}'(v) + \frac{(n-1)\hat{b}(v)f(v)}{F(v)} = \frac{(n-1)vf(v)}{F(v)}$$

Solving this first order differential condition gives

$$\hat{b}(v) = \frac{(n-1)\int_0^v xf(x)F(x)^{n-2}dx}{F(v)^{n-1}} + \frac{A}{F(v)^{n-1}}$$

where A is a constant of integration. If negative bids are excluded, as $v \to 0$, we expect $b(v) \to 0$, thus $A = 0$.

$$\hat{b}(v)_{FP} = \frac{(n-1)\int_0^v xf(x)F(x)^{n-2}dx}{F(v)^{n-1}}$$

Integrating by parts gives $\hat{b}(v)$ in a more convenient expression

$$\hat{b}(v)_{FP} = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \tag{3}$$

Thus we have found a bidding strategy that maximises each bidder's profits for any value v. It follows from the definition of Bayes-Nash equilibrium that the outcome of each bidder bidding $\hat{b}(v_i)$ is a symmetric equilibrium.

The seller's expected revenue (R_{FP}) is the expected highest bid.

$$\begin{aligned} R_{FP} &= E[max\{\hat{b}(v_1), ..., \hat{b}(v_n)\}] \\ &= E[\hat{b}(max\{(v_1), ..., (v_n)\}] \\ &= \int_0^{\overline{v}} n\hat{b}(v)f(v)F(v)^{n-1}dv \end{aligned}$$

Example:

If $v_1, ..., v_n \sim iid$ U(0,1), the bidding strategy is given by

$$\begin{aligned} \hat{b}(v)_{FP} &= \frac{(n-1)\int_0^v x^{n-1}dx}{v^{n-1}} \\ &= \frac{(n-1)v}{n} \end{aligned}$$

Therefore the seller's expected revenue is

$$\begin{aligned} R_{FP} &= \int_0^1 (n-1)v^n dv \\ &= \frac{n-1}{n+1} \end{aligned} \tag{4}$$

Note that as $n \to \infty$, $\hat{b}(v) \to v$ and $\pi_{winner} \to 0$. This can be shown true for any $F(v)$ (see appendix A) and the economic idea behind it is simple: as the number of participants in the auction increases, bidders will have less incentive to shade their bids, their informational rent decreases and bidders will bid their true value.

4.1.2 Second price auction

In this auction the winner pays the highest unsuccessful bid. Placing a bid determines only whether the bidder wins, not the actual payment, and this leads to an *incentive compatible* outcome – bidders signal their true valuations by biding while still maximising expected profit. This is shown formally by applying a similar approach to the first price auction. Consider bidder one's profits for making bid b_1. By (1) the expected profit is

$$\begin{aligned}\pi_1 &= E[(v_1 - max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}I_{\{1\ successful\}}] \\ &= E[(v_1 - X)I_{\{1\ successful\}}]\end{aligned}$$

It is easy to show using order statistics that X has distribution function $F(x)^{n-1}$. Therefore

$$\pi_1 = \int_0^{b_1} (v - x)(n-1)f(x)F(x)^{n-2}dx$$

Bidder one's maximisation problem is to choose $b_1$that maximises π_1, thus our first order condition is

$$\frac{\partial \pi_1}{\partial b_1} = \frac{\partial}{\partial b_1}\int_0^{b_1} (v - x)(n-1)f(x)F(x)^{n-2}dx$$

Applying Leibniz's rule gives

$$\begin{aligned}\frac{\partial \pi_1}{\partial b_1} &= \int_0^{b_1} \frac{\partial}{\partial b_1}[(v_1 - x)(n-1)f(x)F(x)^{n-2}]dx + [(v_1 - b_1)(n-1)f(b_1)F(b_1)^{n-2} \\ &= (v_1 - b_1)(n-1)f(b_1)F(b_1)^{n-2}\end{aligned}$$

Since $F(b) \geq 0$ and $f(b) \geq 0$ for $b \in [0, \bar{b}]$, the solution $v_1 = b_1$is either a maximum or a minimum of π_1.

$$\frac{\partial^2 \pi(b)}{\partial b^2} = (v-b)(n-1)[(n-2)f(b)F(b)^{n-3} + f'(b)F(b)^{n-2}] - (n-1)f(b)F(b)^{n-2}$$

With $v = b$,

$$\frac{\partial^2 \pi(b)}{\partial b^2} = -(n-1)F(v)^{n-2}f(v) < 0$$

Hence the bidding strategy $\hat{b} = v$ maximises π, i.e. bidding your true valuation v is the equilibrium bidding strategy.

The expected seller's revenue from this auction is simply the second highest expected bid. An important result in order-statistics is that the probability density of the rth ordered variate of a sample of n identical and independently distributed variates $v_1, .., v_n$ is

$$f_{v_r}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r}f(x) \tag{5}$$

where $F(x)$ is the distribution function of v. Here we are interested in the probability of the second highest value being below x, thus the $(n-1)th$ density given by (5).

$$\begin{aligned} f_{v_{n-1}}(v) &= \frac{n!}{(n-2)!(n-n+1)!}[F(x)]^{n-2}[1-F(x)]^{n-n+1}f(x) \\ &= n(n-1)[F(x)]^{n-2}[1-F(x)]f(x) \end{aligned}$$

Therefore

$$R_{SP} = \int_0^{\overline{v}} vn(n-1)F(v)^{n-2}[1-F(v)]f(v)dv \tag{6}$$

Example: If $v_1, ..., v_n \sim iid$ U(0,1), the expected seller's revenue is given by

$$\begin{aligned} R_{SP} &= \int_0^1 vn(n-1)v^{n-2}[1-v]dv \\ &= \int_0^1 n(n-1)v^{n-1} - n(n-1)v^n dv \\ &= (n-1) - \frac{n(n-1)}{n+1} \\ &= \frac{n-1}{n+1} \end{aligned} \tag{7}$$

Note that (7) is the same as (4) in a first price auction. In fact this result holds for any distribution $F(v)$ and can be shown as follows:

$$R_{FP} = \int_0^{\overline{v}} n\hat{b}(v)f(v)F(v)^{n-1}dv$$

Substituting $\hat{b}(v) = \frac{(n-1)\int_0^v xf(x)F(x)^{n-2}dx}{F(v)^{n-1}}$ gives

$$R_{FP} = \int_0^{\overline{v}} n(n-1)\int_0^v xf(x)F(x)^{n-2}dxf(v)dv$$

Changing the order of integration (since$0 < x < v$ and $0 < v < \overline{v}$, $\int_x^{\overline{v}} f(v)dv = 1 - F(x)$):

$$R_{FP} = \int_0^{\overline{v}} n(n-1)xf(x)F(x)^{n-2}[1-F(x)]dx = R_{SP} \tag{8}$$

4.1.3 All-pay auction

As the name suggests, in this auction all bidders pay their bid regardless of whether they win. Appealing to our intuition, it seems sensible that bidders will bid more conservatively than in a first price auction: they know they are guaranteed a loss if they do not win, thus will optimise to *minimise* that loss. While the all-pay auction is uncommon in market settings, it has been used to model the arms race (O'Neill, 1986) and rent-seeking activity , such as lobbying (Baye and Kovenock, 1993).

Now for the formal analysis. The payoffs for bidder one are

$$\pi_1 = \begin{cases} v_1 - b_1 & \text{if } b_1 > \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \\ -b_1 & \text{if } b_1 \leq \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \end{cases}$$

Assume that in case of a draw, no one wins the object. The expected profit is,

$$\pi_1 = (v_1 - b_1)Pr(b_1 > max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}) - b_1(1 - Pr(b_1 > max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}))$$

Recall that we can parameterise $\pi_1 = \pi(x)$ so that $Pr(\hat{b}(x) > max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}) = Pr(x > max\{v_2, ..., v_n\}) = F(x)^{n-1}$. Thus

$$\begin{aligned} \pi(x) &= (v - \hat{b}(x))F(x)^{n-1} - \hat{b}(x)(1 - F(x)^{n-1}) \\ &= vF(x)^{n-1} - \hat{b}(x) \end{aligned}$$

Clearly for bidders to make a net gain, $vF(x)^{n-1} > \hat{b}(x)$. Since $0 \leq F(x) \leq 1$, bidders will bid significantly lower amounts as the bidder participation increases. Maximising $\pi(x)$with respect to x gives

$$\frac{\partial \pi(x)}{\partial x} = v(n-1)f(x)F(x)^{n-2} - \hat{b}'(x)$$

In a symmetric equilibrium, the expected profit is maximised at $x = v$. Thus the first order condition is $\pi'(v) = 0$.

$$\begin{aligned} \hat{b}'(v) &= v(n-1)f(v)F(v)^{n-2} \\ \therefore \hat{b}(v) &= \int_0^v x(n-1)f(x)F(x)^{n-2}dx \\ &= vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx + C \end{aligned}$$

The initial value condition is $b(0) = 0$, therefore $C = 0$. Hence we have

$$\hat{b}(v)_{AP} = vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx$$

Recall that the bidding strategy for a first price auction is

$$\hat{b}(v)_{FP} = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}}$$

therefore $\hat{b}(v) = b(v)_{FP}F(v)^{n-1}$. Since $0 < F(v)^{n-1} < 1$, an all-pays auction will always result in lower biding compared to a first price auction. Furthermore, the lower a bidder's valuation is the greater the inequality $\hat{b}(v)_{FP} > \hat{b}(v)_{AP}$.

Example:

If $v_1, ..., v_n \sim$ *iid* U(0,1), the bidding strategy is given by

$$\begin{aligned} \hat{b}(v) &= v^n - \int_0^v x^{n-1}dx \\ &= \frac{n-1}{n}v^n \end{aligned}$$

The case of standard uniform values in a first-price auction gave $\hat{b}(v)_{FP} = \frac{n-1}{n}v > \frac{n-1}{n}v^n$ since $v \in (0,1)$

Since every bidder pays their bid regardless of the auction outcome, the seller's revenue is

$$\begin{aligned} R_{AP} &= \sum_{i=1}^{n} \hat{b}(v_i) \\ \therefore E[R] &= nE[b(V)] \quad \text{since } v_1, ..., v_n \ iid \ V \\ &= n\int_0^{\bar{v}} \hat{b}(v)f(v)dv \\ &= n\int_0^{\bar{v}} vF(v)^{n-1}f(v) - n\int_0^{\bar{v}} \left(\int_0^v F(x)^{n-1}dx\right) f(v)dv \\ &= \overline{v}F(\overline{v})^n - \int_0^{\bar{v}} F(v)^n dv - n\int_0^{\bar{v}} \left(\int_0^v F(x)^{n-1}dx\right) f(v)dv \end{aligned}$$

Changing the order of integration in the last integral (since $\int_x^{\bar{v}} f(v)dv = (1 - F(y))$):

$$\begin{aligned} R_{AP} &= \overline{v} - \int_0^{\bar{v}} F(v)^n dv - n\int_0^{\bar{v}} F(x)^{n-1}(1 - F(x))dx \\ &= \overline{v} + \int_0^{\bar{v}} (n-1)F(v)^n - nF(v)^{n-1}dv \end{aligned}$$

This is the same revenue as in a first and second price auction. To see this, integrating (8) by parts gives:

$$R_{SP} = \int_0^{\bar{v}} n(n-1)vf(v)F(v)^{n-2}[1 - F(v)]dv$$

$$
\begin{aligned}
&= \int_0^{\bar{v}} v\left(n(n-1)f(v)F(v)^{n-2} - n(n-1)f(v)F(v)^{n-1}\right)dv \\
&= v\left(nF(v)^{n-1} - (n-1)F(v)^n\right)\Big|_0^{\bar{v}} - \int_0^{\bar{v}} nF(v)^{n-1} - (n-1)F(v)^n dv \\
&= \bar{v} + \int_0^{\bar{v}} (n-1)F(v)^n dv - nF(v)^{n-1}dv \\
&= R_{AP}
\end{aligned}
$$

What makes the all-pay auction radically different from either first or second-price auctions is that as bidder participation increases, the winner's profit increases. The profit of the winning bidder is

$$\pi_{winner} = v - vF(v)^{n-1} + \int_0^v F(x)^{n-1}dx$$

Clearly as $n \to \infty$, $\pi_{winner} \to v$. Again, after some consideration, this is not particularly surprising. Since seller revenue is no higher than in the first-price auction, the process of taxing all bidders must have a redistributive effect to make someone else better off - here it is the winner.

4.1.4 War of Attrition

As an analogy, the war of attrition is the second-price version of all-pay. The payment system is that all bidders pay their bid except from the winner who pays the second highest (unsuccessful) bid. Again, this auction is rarely seen in the market-place for selling goods, however it has useful modelling properties and it's list of applications includes conflict between insects (Bishop, D., 1978) and the struggle for survival among firms (Fudenburg, D. et al., 1976). Now consider the payoffs for player one

$$\pi_1 = \begin{cases} v_1 - max_{1\neq i}(\hat{b}_i) & \text{if } b_1 > \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \\ -b_1 & \text{if } b_1 \leq \max\{\hat{b}(v_2), ..., \hat{b}(v_n)\} \end{cases}$$

Thus the expected profit is

$$\pi_1 = E[(v_1 - X)I_{\{1\ successful\}}] - (1 - Pr(b_1 > max\{\hat{b}(v_2), ..., \hat{b}(v_n)\}))b_1 \quad (9)$$

Recall that

$$
\begin{aligned}
Pr(successful) &= Pr(b_1 > \hat{b}(v_2), ..., b_1 > \hat{b}(v_n)\} \\
&= Pr(\hat{b}^{-1}(b_1) > v_2, ..., \hat{b}^{-1}(b_1) > v_n\} \\
&= F(\hat{b}^{-1}(b_1))^{n-1}
\end{aligned}
$$

Hence

$$\pi_1 = \int_0^{\hat{b}^{-1}(b)} (v - \hat{b}(x))(n-1)f(x)F(x)^{n-2}dx - (1 - F(\hat{b}^{-1}(b))^{n-1})b \quad (10)$$

Maximising (10) with respect to b yields the first order condition:

$$\begin{aligned}
\frac{\partial \pi}{\partial b} &= (v - \hat{b}(\hat{b}^{-1}(b)))(n-1)f(\hat{b}^{-1}(b))F(\hat{b}^{-1}(b))^{n-2}\frac{1}{\hat{b}'(\hat{b}^{-1}(b))} - (1 - F(\hat{b}^{-1}(b))^{n-1}) \\
&\quad + b(n-1)f(\hat{b}^{-1}(b))F(\hat{b}^{-1}(b))^{n-2} \\
&= v(n-1)f(\hat{b}^{-1}(b))F(\hat{b}^{-1}(b))^{n-2}\frac{1}{\hat{b}'(\hat{b}^{-1}(b))} - (1 - F(\hat{b}^{-1}(b))^{n-1})
\end{aligned}$$

At symmetric equilibrium, $b = \hat{b}(v)$ and $\frac{\partial \pi}{\partial b} = 0$, thus the first order condition is

$$\begin{aligned}
0 &= v(n-1)f(v)F(v)^{n-2}\frac{1}{\hat{b}'(v)} - (1 - F(v)^{n-1}) \\
\hat{b}'(v) &= \frac{v(n-1)f(v)F(v)^{n-2}}{1 - F(v)^{n-1}} \\
\Rightarrow \hat{b}(v)_{WA} &= \int_0^v \frac{x(n-1)f(x)F(x)^{n-2}}{1 - F(x)^{n-1}}dx
\end{aligned}$$

Example: if $v_1, ..., v_n \sim$ *iid* U(0,1), the bidding strategy is given by

$$\hat{b}(v) = \int_0^v \frac{(n-1)x^{n-1}}{1 - x^{n-1}}dx$$

The integral diverges to infinity as $v \to 1$ and there is no general expression of $\hat{b}(v)$ for arbitrary n. So failing to have an exact solution, numerical approximations to $\hat{b}(v)$ are shown below.

n	$v = 0.3$	$v = 0.6$	$v = 0.9$
2	0.0566749	0.316291	1.40259
3	0.0190392	0.186294	1.14444
5	0.0019528	0.0671334	0.810069
10	5.31447×10^{-6}	0.00547102	0.401839
20	3.31245×10^{-11}	0.0000347	0.124309
50	7.0354×10^{-27}	7.9211×10^{-12}	0.005065

Table 1

Increased bidder participation results in more conservative bidding and for very high v it rational to bid more than v, and indeed if $v = 1$, to bid an infinite amount. This is a surprising departure in bidding strategy compared to the previous auctions.

To calculate the expected seller's revenue in this auction, the method employed in the all-pay auction is used. Since all bidders their bid, except the winner who pays the second highest bid, we have

$$R_{WA} = nE[b(V)] - E[b(v_{highest}) + E[b(v_{second\ highest})$$

Recall that the highest expected bid is given by $\int_0^{\bar{v}} n\hat{b}(v)f(v)F(v)^{n-1}dv$. This is simply the revenue from a first price auction. Using the results from a second price auction, the second highest expected bid is

$$E[b(v_{second\ highest}) = \int_0^{\bar{v}} n(n-1)\hat{b}(v)F(v)^{n-2}[1-F(v)]f(v)dv$$

Thus

$$\begin{aligned}
R_{WA} &= n\int_0^{\bar{v}} \hat{b}(v)f(v)dv - \int_0^{\bar{v}} n\hat{b}(v)f(v)F(v)^{n-1}dv + \int_0^{\bar{v}} n(n-1)b(v)F(v)^{n-2}[1-F(v)]f(v)dv \\
&= \int_0^{\bar{v}} n\hat{b}(v)f(v) + n(n-1)\hat{b}(v)F(v)^{n-2}f(v) - n^2\hat{b}(v)F(v)^{n-1}f(v)dv \\
&= \int_0^{\bar{v}} n\hat{b}(v)f(v)\left(1+(n-1)F(v)^{n-2} - nF(v)^{n-1}\right)dv \\
&= \int_0^{\bar{v}} \left(\int_0^{v} \frac{x(n-1)f(x)F(x)^{n-2}}{1-F(x)^{n-1}}dx\right) n\left(f(v)+(n-1)f(v)F(v)^{n-2} - nf(v)F(v)^{n-1}\right)dv
\end{aligned}$$

Since we have

$$\begin{aligned}
\int_x^{\bar{v}} f(v)+(n-1)f(v)F(v)^{n-2} - nf(v)F(v)^{n-1}dv &= \left(F(v)+F(v)^{n-1}-F(v)^n\right)\Big|_x^{\bar{v}} \\
&= 1-F(x)-F(x)^{n-1}+F(x)^n
\end{aligned}$$

changing the order of integration gives:

$$\begin{aligned}
R_{WA} &= \int_0^{\bar{v}} \frac{x(n-1)f(x)F(x)^{n-2}}{1-F(x)^{n-1}} n(1-F(x)-F(x)^{n-1}+F(x)^n)dx \\
&= \int_0^{\bar{v}} \frac{x(n-1)f(x)F(x)^{n-2}}{1-F(x)^{n-1}} n(1-F(x))(1-F(x)^{n-1})dx \\
&= \int_0^{\bar{v}} n(n-1)xf(x)F(x)^{n-2}(1-F(x))dx \\
&= R_{SP}
\end{aligned}$$

4.2 Revenue Equivalence

In the previous section, we see that the seller's expected revenue from first price, second price, all-pay and war of attrition auctions are all equivalent. This is due to the Revenue Equivalence Theorem and is auction theory's most celebrated theorem. It states the conditions which different auction formats yield the same expected revenue.

Revenue Equivalence Theorem (RET) *Assume each of a given number of risk-neutral potential buyers has a privately-known valuation independently*

drawn from a strictly-increasing atomless distribution, and that no buyer wants more than one of the k identical indivisible prizes.

Then any mechanism in which (i) the prizes always go to the k buyers with the highest valuations and (ii) any bidder with the lowest feasible valuation expects zero surplus, yields the same expected revenue (and results in each bidder making the same expected payment as a function of her valuation). (Klemperer, 1999)

In all four of the auctions analysed, each requirement for RET to hold is met. From design, bidders are risk-neutral and have i.i.d. private values. Each auction awards objects to the bidder with highest valuation. To prove this last point, consider each auction's bidding strategy:

$$\begin{aligned}
\hat{b}(v)_{FP} &= v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \\
\hat{b}(v)_{SP} &= v \\
\hat{b}(v)_{AP} &= vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx \\
\hat{b}(v)_{WA} &= \int_0^v \frac{x(n-1)f(x)F(x)^{n-2}}{1-F(x)^{n-1}}dx
\end{aligned}$$

Differentiating with respect to v gives:

$$\begin{aligned}
\hat{b}'(v)_{FP} &= \frac{(n-1)f(v)\int_0^v F(x)^{n-1}dx}{F(v)^n} > 0 \\
\hat{b}'(v)_{SP} &= 1 > 0 \\
\hat{b}'(v)_{AP} &= v(n-1)f(v)F(v)^{n-2} > 0 \\
\hat{b}'(v)_{WA} &= \frac{v(n-1)f(v)F(v)^{n-2}}{1-F(v)^{n-1}} > 0
\end{aligned}$$

Hence bidders with higher than others values submit strictly higher bids and the bidder with highest value submits highest bid.

Once an auction design satisfies RET, the theorem can be used to easily determine equilibrium bidding strategy. See appendix A for an example.

4.3 Reserve pricing

In this section the effect of introducing a reserve price, denoted r, on bidding behaviour and seller's revenue is considered. Our model assumes the reserve price is publicly known by all bidders and the seller only accepts bids above or equal it. There is no chance of the seller "cheating" by removing the reserve price after bids are collected.

4.3.1 First price auction

The analysis here is similar to the normal first price auction. Now the winning bid has the added condition that $b_{winner} \geq r$**.** Thus the payoffs to bidder one are

$$\pi_1 = \begin{cases} v_1 - b_1 & \text{if } b_1 > \max\{\hat{b}(v_2), ..., \hat{b}(v_n), r\} \\ 0 & \text{if } b_1 \leq \max\{\hat{b}(v_2), ..., \hat{b}(v_n), r\} \end{cases}$$

Now that a bidder knows that his or her bid must exceed r to win, there exists a value ρ whereby if $v_i < \rho$, bidder i does not participate and if $v_i \geq \rho$, bidder i participates. For $v = \rho$, the bidder is indifferent to participating, i.e. $\pi_1 = 0$, implying that $(\rho - r)F^{n-1}(\rho) = 0$ where $F^{n-1}(\rho)$ is the probability that an indifferent bidder is the only participating bidder. Hence $\rho = r$, and $b(r) = r$. Using this we can express expected profits as

$$\pi(x)_1 \quad = \quad (v - \hat{b}(x))F_V(x)^{n-1} \times Pr(\hat{b}(x) \geq r)$$

Possible cases are: $b_1 < r$, $\pi(x)_1 = 0$; if $b_1 = r$, $\pi(x)_1 = (v - r)F_V(x)^{n-1}$; if $b_1 \geq r$, $\pi(x)_1 = (v - b(x))F_V(x)^{n-1}$. Consider the last case. To maximise $\pi(x)_1$, we have

$$\pi'(x) = (n-1)(v - \hat{b}(x))f(x)F(x)^{n-2} - \hat{b}'(x)F(x)^{n-1}$$

In symmetric equilibrium π is maximised at $x = v$, hence the first order condition is $\pi(v)' = 0$

$$\hat{b}'(v) + \frac{(n-1)\hat{b}(v)f(v)}{F(v)} \quad = \quad \frac{(n-1)vf(v)}{F(v)}$$

$$\begin{aligned} \therefore \hat{b}(v) \quad &= \quad \frac{(n-1)\displaystyle\int_r^v xf(x)F(x)^{n-2}dx}{F(v)^{n-1}} + A \\ &= \quad v - r - \frac{\displaystyle\int_r^v F(x)^{n-1}dx}{F(v)^{n-1}} + A \end{aligned}$$

where A is a constant of integration. When $v = r$, we have that $\hat{b}(r) = r$. Thus $A = r$ and the equilibrium bidding strategy for $v \geq r$ is

$$\hat{b}(v)_{FP} = v - \frac{\displaystyle\int_r^v F(x)^{n-1}dx}{F(v)^{n-1}}$$

The expected seller's revenue is

$$R_{FP} = \int_r^{\overline{v}} n\hat{b}(v)f(v)F(v)^{n-1}dv$$

Comparing the seller's revenue here to the case of no reserve price, we see that the integral is now evaluated from r to $\overline{v}$, reflecting the new initial-value condition with reserve pricing. Substituting in $\hat{b}(v)$ gives

$$\begin{aligned}
R_{FP} &= \int_r^{\overline{v}} n\left(v - \frac{\int_r^v F(x)^{n-1}dx}{F(v)^{n-1}}\right) f(v)F(v)^{n-1}dv \\
&= \int_r^{\overline{v}} nvf(v)F(v)^{n-1} - f(v)\int_r^v F(x)^{n-1}dxdv \\
&= \int_r^{\overline{v}} nvf(v)F(v)^{n-1}dv - \int_r^{\overline{v}}\left(\int_r^v F(x)^{n-1}dx\right) f(v)dv
\end{aligned}$$

Changing the order of integration in the last integral gives

$$\begin{aligned}
R_{FP} &= \int_r^{\overline{v}} nvf(v)F(v)^{n-1}dv - \int_r^{\overline{v}} n(1-F(v))F(v)^{n-1}dv \\
&= vF(v)^n|_r^{\overline{v}} + (n-1)\int_r^{\overline{v}} F(v)^n dv - n\int_r^{\overline{v}} F(v)^{n-1}dv \qquad (11)
\end{aligned}$$

Example If $v_1, ..., v_n \sim$ *iid* U(0,1) and $r \in [0,1]$, the bidding strategy is

$$\begin{aligned}
\hat{b}(v) &= v - \frac{\int_r^v x^{n-1}dx}{v^{n-1}} \\
&= v - \frac{v}{n} + \frac{r^n}{nv^{n-1}}
\end{aligned}$$

Notice that using a reserve price now makes bidding more *aggressive*: now at each v, bidders bid a higher fraction of their value. Now lets calculate R.

$$\begin{aligned}
R_{FP} &= \int_r^1 n\left(\frac{n-1}{n}v + \frac{r^n}{nv^{n-1}}\right) v^{n-1}dv \\
&= \int_r^1 (n-1)v^n + r^n dv \\
&= \left(\frac{(n-1)v^{n+1}}{n+1} + r^n v\right)\Bigg|_r^1 \\
&= \frac{(n-1)}{n+1} + r^n - \frac{(n-1)r^{n+1}}{n+1} - r^{n+1} \\
&= \frac{(n-1)}{n+1} + \left(\left(\frac{2}{n+1} - 2\right)r + 1\right) r^n
\end{aligned}$$

Note that $\left(\frac{2}{n+1} - 2\right)r + 1 = 0$ where $r = \frac{-1}{2/(n+1)-2}$. For r less than $\frac{-1}{2/(n+1)-2}$, expected revenue with the reserve price is higher than with $r = 0$. Let us find

the optimum r:

$$\frac{d}{dr}\left(r^n - \frac{(n-1)r^{n+1}}{n+1} - r^{n+1}\right) = \left(\frac{n}{r} - 2r\right) r^n = 0 \tag{12}$$

Solving (12) gives $r = 1/2$. Generally, the optimum r satisfies (see appendix A for proof):

$$r = \frac{1 - F(r)}{f(r)}$$

Thus the optimal r is strictly positive - it is never optimal to set a zero reserve price.

4.3.2 Second price auction

Introducing a reserve price r in a second price auction does not remove incentive compatibility: that bidding $b = v$ is the dominant strategy. To see this, consider the profits of bidder one:

$$\pi_1 = E[(v_1 - X)I_{\{1\ successful\}}]$$

where $I_{\{1\ successful\}} = I_{b_1 > max(b(v_2),...,b(v_n),r)}$ and $X = max\{b(v_2), ..., b(v_n), r\}$. Consider the two cases: $max(v_2, ..., v_n) < r$ and $max(v_2, ..., v_n) \geq r$. The first happens with probability $F(r)^{n-1}$,the other with probability $1 - F(r)^{n-1}$. By the law of total probability,

$$\begin{aligned}
\pi_1(b_1) &= E[(v_1 - r)I_{\{v_1 > r\}}]F(r)^{n-1} + E[(v_1 - max_{1 \neq i}\{b(v_i)\})I_{\{b_1 > max(b_i)\}}](1 - F(r)^{n-1}) \\
&= (v_1 - r)F(r)^n + \int_r^{b_1} (v - x)(n-1)f(x)F(x)^{n-2}dx(1 - F(r)^{n-1})
\end{aligned}$$

After applying Leibniz's rule, we have

$$(v_1 - b_1)(n-1)f(b_1)F(b_1)^{n-2} = 0,\ v \geq r$$

Thus the equilibrium bidding strategy is to bid $b(v) = v$ for $v \geq r$ and not to bid for $v < r$. Now lets compute the seller's expected revenue.

$$\begin{aligned}
R_{SP} &= \int_r^{\overline{v}} vn(n-1)F(v)^{n-2}[1 - F(v)]f(v)dv + nrF(r)^{n-1}(1 - F(r)) \\
&= \int_r^{\overline{v}} v\left(n(n-1)F(v)^{n-2}f(v) - n(n-1)F(v)^{n-1}f(v)\right)dv + nrF(r)^{n-1}(1 - F(r)) \\
&= v\left(nF(v)^{n-1} - (n-1)F(v)^n\right)\Big|_r^{\overline{v}} - \int_r^{\overline{v}} nF(v)^{n-1} - (n-1)F(v)^n dv + nrF(r)^{n-1}(1 - F(r)) \\
&= \overline{v} - rF(r)^n + \int_r^{\overline{v}} (n-1)F(v)^n - nF(v)^{n-1}dv \qquad (13)
\end{aligned}$$

Note that (13) and (11) are the same, and that we have revenue equivalence. This is not surprising since the bidder with highest valuation will still win the auction with reserve pricing, thus the requirements of RET are met. Also note that the optimal reserve price is still $r = \frac{1-F(r)}{f(r)}$ in a second price auction.

5 Conclusion

The basic game-theoretic model presented in this paper gives rise to a number of important results: formal equilibrium bidding strategy, the revenue equivalence theorem and the optimal design for auctions with reserve prices. The model is a solid foundations for further development in auction theory.

6 Appendix A

6.1 Proof that $b(v) \to v$ as $n \to \infty$ in a first price auction

Recall that the equilibrium bidding strategy is

$$\hat{b}(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx \tag{14}$$

Since F is an increasing function, except at point v, we have that $\frac{F(x)}{F(v)} < 1$ for all x in $[0, v]$. Thus we expect the integral in (14) to converge to 0 and the bidding function to converge to v. This result can be shown formally by proceeding in the following way. Let $\varepsilon > 0$. For all v, we may write the bidding strategy as

$$\hat{b}(v) = v - \int_0^{v-\varepsilon/2} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx - \int_{v+\varepsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx$$

We have,

$$\left|\int_{v+\varepsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| \leq \int_{v+\varepsilon/2}^{v} \left|\left(\frac{F(x)}{F(v)}\right)^{n-1}\right| dx \leq \int_{v+\varepsilon/2}^{v} 1 dx \leq \varepsilon/2 \quad \text{and}$$

$$\left|\int_0^{v-\varepsilon/2} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| \leq \int_0^{v-\varepsilon/2} \left|\left(\frac{F(v-\varepsilon/2)}{F(v)}\right)^{n-1}\right| dx \leq \int_0^{1} \left|\left(\frac{F(v-\varepsilon/2)}{F(v)}\right)^{n-1}\right| dx$$

Since $\frac{F(v-\varepsilon/2)}{F(v)} < 1$, $\left(\frac{F(v-\varepsilon/2)}{F(v)}\right)^{n-1} \to 0$ as $n \to +\infty$ so that

$$\exists\ n_0\ s.t\ \left|\left(\frac{F(v-\varepsilon/2)}{F(v)}\right)^{n-1}\right| \leq \varepsilon/2 \text{ for all } n > n_0$$

Therefore, for all $n > n_0$, we have

$$\left|\hat{b}(v) - v\right| \leq \left|\int_{v+\varepsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| + \left|\int_0^{v-\varepsilon/2} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right|$$

$$\therefore \left|\hat{b}(v) - v\right| \quad \leq \quad \varepsilon/2 + \varepsilon/2$$

6.2 Proof that $r = \frac{1-F(r)}{f(r)}$ is the optimal reserve price in first price auctions

We have that $\int_r^{\overline{v}} vn(n-1)F(v)^{n-2}[1-F(v)]f(v)dv$

$$R_{FP}(r) = \int_r^{\overline{v}} nvf(v)F(v)^{n-1}dv - \int_r^{\overline{v}} n(1-F(v))F(v)^{n-1}dv$$

Applying Leibniz's rule gives

$$R'_{FP}(r) = -nrf(r)F(r)^{n-1} + n(1-F(r))F(r)^{n-1}$$

Thus at the optimal r,

$$\begin{aligned} nrf(r)F(r)^{n-1} &= n(1-F(r))F(r)^{n-1} \\ r &= \frac{1-F(r)}{f(r)} \end{aligned}$$

6.3 Using RET to find equilibrium bidding strategy

I will use the first-price auction as an example. Recall that the expected payment of bidder with value v is just $Pr(v > max_{v \neq v_j} v_j)$ times the expectation of the highest of the remaining $(n-1)$ values conditional on all these values being less than v. The density of the highest $(n-1)$ values is $(n-1)f(v)F(v)^{n-2}$, and the last expectation can be written as

$$\frac{\int_0^v x(n-1)f(x)F(x)^{n-2}dx}{\int_0^v (n-1)f(x)F(x)^{n-2}dx}$$

Integrating the numerator and denominator by parts gives

$$\begin{aligned} &= \frac{vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \\ &= v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \end{aligned}$$

Since in a first price the expected payment for a bidder is their bid times $Pr(v > max_{v \neq v_j} v_j)$, it follows that

$$\widehat{b}(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}}.$$

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