A Primer in Financial Theory for Sam

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1 Prices and Returns

Let P_t be the price of a traded asset. Trading is assumed to occur at discrete times $t = \{0, t_1, t_2, ...\}$. The *average return* on the asset is defined by

$$R_t^{(k)} \equiv \frac{P_t - P_{t-k}}{P_{t-k}},$$

and daily returns are defined with $k = 1$. An important, almost equivalent definition is

$$r_t^{(k)} \equiv \log(P_t) - \log(P_{t-k}).$$

Why is this almost equivalent? Do a Taylor expansion on $x = (P_t/P_{t-k})$ centered at 1:

$$\begin{aligned} r_t^{(k)} &= \sum_{j=0}^{\infty} \frac{f^{(j)}(1)(x-1)^j}{j!} \\ &= 0 + \frac{1}{1}\left(\frac{P_t}{P_{t-k}} - 1\right) + o(t^2) \\ &= R_t^{(k)} + o(t^2) \end{aligned}$$

In continuous time, returns are best defined though rate r_t, such that the average return is

$$R_t^{(k)} = \int_{t-k}^{t} r_t dt$$

2 Some Price Models

Random Walk Hypothesis Under strongly efficient markets, the following model might hold:

$$r_t = \mu + \varepsilon_t, \ \ \varepsilon_t \sim (0, \sigma^2),$$

where ε_t are i.i.d fluctuations (e.g. trader noise, information effects), μ is the mean daily return, σ is the daily stock volatility, and both are constant. Applying the log definition of returns, we have

$$\begin{aligned} \log(P_t/P_{t-1}) &= \mu + \varepsilon_t \\ \Rightarrow P_t &= P_{t-1}e^{\mu+\varepsilon_t} \\ \Rightarrow P_t &= P_0 e^{\mu t + \sum \varepsilon_t} \end{aligned}$$

Thus, $P_t = P_0 e^{\mu t + X_t}, \ X_t \sim (0, \sigma^2 t)$. If $X_t \sim N(0, \sigma^2 t)$, the expected price is

$$E[P_t|P_0] = P_0 e^{(\mu+\sigma^2/2)t}.$$

Without making distributional assumptions, we see that prices increase exponentially with local random walk behaviour, and the variance in prices increases exponentially in time.

If μ_t and σ_t are not constant, we have the following model solution,

$$P_t = P_0 e^{\int_0^t \mu_t dt + \sum \varepsilon_t}.$$

Both these models are discrete time. The equivalent model in *continuous* time is specified by

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t,$$

where $W_t \sim N(0, \sigma t)$ is a Brownian motion, a fundamental type of random walk, and$dW_t = \lim_h (W_t - W_{t-h})$. This is the price model used in the celebrated Black-Scholes equation, and is important but unrealistic. The solution is

$$P_t = P_0 e^{\int_0^t (\mu_t - \frac{\sigma^2}{2})dt + X_t}, \ \ X_t \sim N(0, \sigma_t^2 t).$$

ARCH and GARCH Models These are widely used models for financial returns developed by Engel, 1981. The basic ARCH(1) model is

$$\sigma_t^2 = \alpha\sigma_{t-1}^2 + \varepsilon_t, \ \ r_t = \mu + \sigma_t z_t,$$

where $0 < \alpha < 1$, ε_t and z_t are i.i.d error terms. Here, volatility is assumed to follow an autoregressive (AR) process of order 1. Returns are independent of each other, but are conditionally heteroskedastic. A generalization is the GARCH(1,1) model:

$$\sigma_t^2 = \alpha\sigma_{t-1}^2 + \beta\varepsilon_{t-1} + \varepsilon_t, \ \ r_t = \mu + \sigma_t z_t.$$

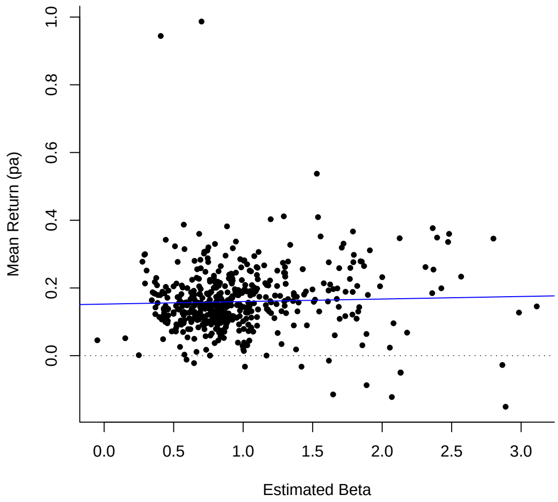
Here, volatility follows an autoregressive-moving average process, ARMA(1,1).

3 Some Results

The CAPM For every stock i with return $r_{t,i}$, we have

$$Er_{t,i} - r_f = \beta_i E(r_{t,m} - r_f)$$

where $\beta_i = \frac{Cov(r_{t,m}, r_{t,i})}{Var(r_{t,m})}$, r_f is the risk-free rate (e.g. 1-year T-bill rate), and $r_{t,m}$ is the return on the "market portfolio". Rolls' critique: no such portfolio exists in any measurable sense. Most studies use a value-weighted portfolio, like the S&P500 index, or MSCI World Index. Below is a look at how the CAPM fares. There should be a significant linear response with slope equal to $Er_m - r_f$. CAPM is rejected in this sample.



Put-Call Parity For a put and call with equal strike price K, and maturity T, the value of a put P_t and call C_t is related by

$$C_t + Ke^{-r(T-t)} = P_t + S_t$$

where r is the risk free rate, $S_t$is the stock price at t. This holds pretty well against empirical testing.

4 How to Analyse Financial Time Series

Nearly all raw financial time series are *non-stationary*. Examples include stock prices, exchange rates, interest rates. These need to be transformed such they are stationary, meaning that the processes have a well defined mean, variance and covariance structure that is constant in time.

The usual trick is to *difference* the series, diff(...) in *R*, in the case of exchange and interest rates, and to log-difference the series for prices, i.e. deal with *returns.*

Things to look for in stationary series:

- The distribution: are returns distributed normally? Usually not.

- What is the correlation structure between adjacent measurements? An autocorrelation plot does this, use acf(...) in *R*.

- What is the correlation structure between *absolute* of measurements? You'll see something interesting. Its called long-range dependence.

- Is there cross-stocks (or better, cross-markets) correlation? Use the ccf(...) function.

5 Variance Bound Theory

The Miller and Modigliani (1961) Formula The fundamental stock price for infinite holding period with a constant interest rate is

$$P_t = \sum_{\tau=1}^{\infty} \frac{E_t(d_{t+\tau})}{(1+r)^{\tau}},$$

where $d_{t+\tau}$ is the dividend payed at $t+\tau$, r is the risk free rate. *Proof:* Suppose not, show arbitrage exists under rational expectations. For finite holding times, until T, this becomes

$$P_t = \sum_{k=1}^{T-t} \frac{E_t(d_{t+k})}{(1+r)^k} + \frac{E_t(P_T)}{(1+r)^{T-t}}.$$

Under perfect foresight, we have

$$P_t^* = \sum_{k=1}^{T-t} \frac{d_{t+k}}{(1+r)^k} + \frac{P_T}{(1+r)^{T-t}}, \qquad (1)$$

which implies

$$EP_t^* = P_t, \text{ or } P_t^* = P_t + \varepsilon_t$$

where $E(\varepsilon_t) = 0$, ε_t independent of P_t (*proof:* rational expectations). Applying $Var(.)$ on both sides yields

$$Var(P_t^*) > Var(P_t) \quad (!)$$

How you calculate P_t^*? Can't use the infinite holding period formula since we do not know future dividends for ever. Instead, can calculate past P_t^* using today's P_T using (1). Set $P_T^* = P_T$, back calculate. For the first paper on this area, see Shiller (1981). He makes a bad mistake in estimating $Var(P_t)$ and $Var(P_t^*)$ – these are *nonstationary* random variables, so you cannot look at the raw S^2 values for price, instead look at the variance of returns and aggregate under some model, e.g. random walk model.