A Complete Derivation of the Capital Asset Pricing Model

Matthew Pollard

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This note presents four derivations of the Capital Asset Pricing Model pricing result under different sets of assumptions. The pricing result is that for all assets i and all periods t,

$$E_t[R_{i,t+1}] - R_f = \beta E_t[R_{m,t+1} - R_f]$$

where $R_{m,t+1}$ is the market return and R_f is the risk-free asset.

There are five sets of model assumptions that lead to the pricing result. These are:

A1 : The market is mean-variance efficient.

A2 : All returns are conditionally normal; investors maximise expected utility.

A3 : Investors have quadratic utility functions and maximise expected utility.

A4 : Investors have stochastic discount factors perfectly correlated to the the market return and maximise expected utility.

I show that market mean-variance efficiency (A1) is a sufficient and necessary condition for the CAPM to hold – the statements are equivalent. The subsequent model assumptions (A2-A4) each lead to models where all investors are *effective* mean-variance maximizers, or in other words, in equilibrium they do not care about any other central moment of their portfolios. The consequence is that each hold mean-variance efficient portfolios; the market is mean-variance efficient and therefore the CAPM holds.

1 Mean-Variance Efficiency

Claim Let x_p denote any mean-variance efficient portfolio with expected return $E_t[R_{p,t+1}]$. Then for all assets i,

$$E_t[R_{i,t+1}] - R_f = \beta E_t[R_{p,t+1} - R_f]$$

Furthermore, any portfolio that satisfies the above conditions is necessarily mean-variance efficient.

Proof

Let $\Sigma = (\sigma_{i,j})_{N\times N}$ denote the matrix of asset covariances. Suppose x_p is efficient; that is, it is the solution to the optimization problem

$$x_p = \arg\ \min \frac{1}{2} x^T \Sigma x \text{ subject to } x_p = ER_p,\ \ x_p^T \mathbf{1} = 1.$$

The necessary and sufficient first order conditions for a solution are:

$$\begin{aligned} \Sigma x_p &= \lambda\mu + \kappa\mathbf{1} \\ ER_p &= x_p^{'}\mu \\ 1 &= x_p^{'}\mathbf{1}. \end{aligned} \tag{1}$$

Solving these first order conditions for x_p yields

$$x_p = (\lambda\Sigma^{-1}\mu + \kappa\Sigma^{-1}1), \;\; x_p'\mu = E[R_p]$$

where λ and κ are

$$\lambda = \frac{1}{D}(CE[R_p] - A), \;\; \gamma = \frac{1}{D}\left(B - AE[R_p]\right). \tag{2}$$

and where the scalars (A, B, C, D) are defined as

$$A \;\; = \;\; \mu^{'}\Sigma^{-1}\mathbf{1}, \;\; B = \mu^{'}\Sigma^{-1}\mu, \;\; C = \mathbf{1}\Sigma^{-1}\mathbf{1}, \;\; D = BC - A^2.$$

By definition of covariance,

$$Cov(R_p, R_q) = x_p'\Sigma x_q.$$

for all portfolios x_p and x_q. Suppose x_p is efficient. Using the first order (1) it follows that

$$\begin{aligned} Cov(R_p, R_q) &= (\lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}\mathbf{1})'\Sigma x_q \\ &= \lambda\mu'\Sigma^{-1}\Sigma x_q + \gamma 1'\Sigma^{-1}x_q \\ &= \lambda\mu^{'}x_q + \gamma 1'x_q \\ &= \lambda E[R_q] + \gamma. \end{aligned}$$

Substituting in values of λ and γ yields the expression

$$Cov(R_p, R_q) = \frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}.$$

The risk-free asset neccessarily satisfies $Cov(R_p, R_f) = 0$. Using this fact gives yields the expression for the risk free rate:

$$E[R_f] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}}.$$

The variance of R_p is $Cov(R_p, R_p)$. Dividing the above by $Var(R_p)$ yields

$$\frac{Cov(R_p, R_q)}{Var(R_p)} \;\; = \;\; \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}$$

Isolating $E[R_q]$ and setting $\frac{Cov(R_p,R_q)}{Var(R_p)} = \beta_q$ yields,

$$\begin{aligned} E[R_q] &= \frac{\beta_q Var(R_p) - \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + E[R_{zp}]. \end{aligned}$$

The first term in this epression can be further decomposed:

$$\begin{aligned} \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} &= \beta_q \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} \\ &= \beta_q\left(E[R_p] - \frac{A}{C} + \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}}\right) \\ &= \beta_q(E[R_p] - R_f) \end{aligned}$$

Substituting this this back into the expression $E[R_q]$ yields the CAPM equation

$$E[R_q] = \beta_q(E[R_p] - R_f) + R_f.$$

We have now proven that mean-variance eciency of x_p implies the CAPM equation.

To prove that the CAPM equation implies mean-variance eciency of x_p, the previous algebraic manipulations can be reversed to obtain the necessary and suffcient first order conditions for mean-variance effciency. By suffciency, it immediately follows x_p is mean-variance effcient.

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2 Normally Distributed Returns

Claim If every asset i is normally distributed with known mean and variance $(\mu_{i,t}, \sigma^2_{i,t})$. investors maximise expected utility and are risk-adverse, the CAPM pricing equation holds.

Proof Let the utility function for an investor be denoted $u_i(R_{p,t})$ where $R_{p,t}$ is the stochastic return on portfolio at time t. The investor solves the optimization problem

$$\max_p \; E[u_i(R_{p,t})] \text{ for all } t. \tag{3}$$

The Taylor expansion of the expected utility function about the mean $\mu_{p,t}$ is:

$$\begin{aligned} E[u_i(R_{p,t})] &= E \left[\sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(\mu_{p,t})}{dR^k_{p,t}} (R_{p,t} - \mu_{p,t})^k \right] \\ &= u(\mu_{p,t}) + 0 + u^{(2)}(\mu_{p,t}) E[(R_{p,t} - \mu_{p,t})^2] + u^{(3)} E[(R_{p,t} - \mu_{p,t})^3] \\ &\quad + u^{(4)} E[(R_{p,t} - \mu_{p,t})^4] + u^{(5)} E[(R_{p,t} - \mu_{p,t})^5] + ... \end{aligned}$$

Under the assumption of normality, the central moments can be computed. The odd central moments are zero: $E[(R_{p,t} - \mu_{p,t})^3] = E[(R_{p,t} - \mu_{p,t})^5] = ... = 0$. The even central moments are

$$\begin{aligned} E[(R_{p,t} - \mu_{p,t})^2] &= \sigma^2_{p,t} \\ E[(R_{p,t} - \mu_{p,t})^4] &= 3\sigma^4_{p,t} \\ E[(R_{p,t} - \mu_{p,t})^{2n}] &= \frac{(2n)!}{n! 2^n} \sigma^{2n}_{p,t}. \end{aligned}$$

Importantly, each central moment is a power of $\sigma^2_{p,t}$ and thus we can write

$$Eu(R_{p,t}) = u(\mu_{p,t}) + \sum_{n=1}^{\infty} f(n) u^{(2n)}(\mu_{p,t}) \sigma^{2n}_{p,t}.$$

Under the assumption of risk-adversion, $u^{(2n)}(\mu_{p,t}) < 0$. Therefore, each investor is an *effective* mean-variance maximizer regardless of their utility function. Every investor will hold a mean-variance efficient portfolio to the solution to (3).

The market portfolio is the weighted sum of all investors' portfolios. Let $x^{(k)}_{p,t}$ denote the *kth* investor's mean-variance efficient portfolio; the market portfolio equals

$$x_{m,t} = \sum w^{(k)}_t x^{(k)}_{p,t},$$

where the weights $w_t^{(k)}$ is the relative wealth of each investor as a fraction. A linear combination of mean-variance efficient portfolios is also mean-variance efficient, and hence the market is mean-variance efficient. By the mean-variance/beta equivalence, the CAPM holds.

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Remark The argument can be generalized to the class of *spherically-invariant* distributions, which maintain the central moment property of being parameterized by only the mean and variance.

3 Quadratic Utility

Claim If each investor has a utility function $u_i(R_{p,t}) = a_i R_{p,t} - b_i R_{p,t}^2$ and maximises expected utility, the CAPM pricing equation holds.

Proof The expected utility is

$$Eu_i(R_{p,t}) = a_i \mu_{p,t} - b_i(\sigma_{p,t}^2 + \mu_{p,t}^2),$$

and therefore investors have preferences over only the mean and variance. Each investor holds a mean-variance efficient portfolio. Therefore the market is mean-variance efficient and the CAPM follows.

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4 Linear Stochastic Discount Factor

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Model All investors maximises expected utility of consumption subject to a budget constraint:

$$\max_{c_t} \ \mathbb{E}_t \sum_{\tau=0}^{T} \beta^\tau u_k(C_{t+\tau}) \tag{4}$$

$$\text{subject to } C_t + X_t P_t^i = X_{t-1}(P_t^i + d_t^i)$$

where P_t^i is the price of asset i, X_t is the dollar-amount of each asset bought, d_t^i is the dividend paid over the period t. The return over i is defined as $r_t^i = \frac{P_t + d_t - P_{t-1}}{P_{t-1}}$. The budget constraint imposes

$$C_t = X_{t-1}(P_t^i + d_t^i) - X_t P_t^i, \ \Rightarrow \frac{dC_t}{dX_t} = -P_t^i,$$

The problem in (4) is equivalent to choosing optimal X_t at each t :

$$\max_{X_t} \ \mathbb{E}_t\{u(C_t[X_t]) + \beta u(C_{t+1}[X_{t+1}]) + ...\}$$

Differentiating with respect to X_t and using the chain rule yields the FOC,

$$\mathbb{E}_t\{-u'(C_t)P_t^i + \beta u'(C_{t+1})(P_{t+1}^i + d_{t+1})\} = 0 \tag{5}$$

(Note that terms $C_{t+\tau},\ \tau > 2$ have no X_t terms thus are constant with respect to X_t). Rearranging (2) yields

$$\mathbb{E}_t \left\{ \beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{(P^i_{t+1} + d^i_{t+1})}{P^i_t} \right\} = 1,$$

which is equivalent to

$$\mathbb{E}_t\{m_{t+1} R^i_{t+1}\} = 1 \tag{6}$$

for all assets i and times t, $R^i_t = 1 + r^i_t$, and $m_{t+1} \triangleq \beta \frac{u'(C_{t+1})}{u'(C_t)}$ is called the *stochastic discount factor.*

Claim If m_t is perfectly correlated with the return on the market portfolio, R^m_t, the CAPM holds.

Proof Perfect correlation is equivalent to $m_{t+1} = a + bR^m_{t+1}$. From here forth, all expectations are with respect to information at time $t-1$. We have for all i,

$$\mathbb{E}\{m_t R_{i,t}\} = \mathbb{E}\{(a + bR_{m,t}) R_{i,t}\} = 1$$

In particular, for a portfolio uncorrelated to the market (the risk-free portfolio), we have

$$\mathbb{E}(a + bR_m) r_f = 1 \ \Rightarrow\ r_f = \frac{1}{a + b\mathbb{E}R_m} \tag{7}$$

Using $\mathbb{E}(XY) = Cov(X,Y) + \mathbb{E}(X)\mathbb{E}(Y)$ yields the equation,

$$1 = \mathbb{E}(a + bR_m)R_i = Cov(a + bR_m, R_i) + \mathbb{E}(a + bR_m)\mathbb{E}R_i.$$

Substituting equation (7):

$$b\,Cov(R_m, R_i) + \frac{\mathbb{E}R_i}{r_f} = 1,\ \Leftrightarrow\ \mathbb{E}R_i - r_f = -br_f\,Cov(R_m, R_i). \tag{8}$$

In particular, this holds for $i = m$, and $Cov(R_m, R_m) = Var(R_m)$, thus

$$\mathbb{E}R_m - r_f = -br_f\,Var(R_m).$$

From (8),

$$\begin{aligned} \mathbb{E}R_i - r_f &= \frac{-br_f\,Cov(R_m, R_i)}{-br_f\,Var(R_m)} \times -br_f\,Var(R_m) \\ &= \frac{Cov(R_m, R_i)}{Var(R_m)}\{\mathbb{E}R_m - r_f\} \\ &= \beta_i(\mathbb{E}R_m - r_f) \end{aligned}$$

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