Optimal Selling Method in Several Item Auctions*

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Abstract

This paper studies multi-object auctions where bidders have private and additive values. Three methods of multi-object auctioning are analysed: separate-object, pure bundling and mixed bundling. Equilibrium bidding strategy and seller's expected revenue are formally determined in the case where there are two bidders and the sub-optimality of separate-object auctions compared to pure bundling is shown.

1 Introduction

This paper considers an auction with risk-neutral agents having independent private valuations for several (K) non-identical objects. It considers three different methods of multi-object auctioning when there are two bidders: selling each object in separate sealed-bid Vickery auctions; a single Vickery all-object bundle auction, and a simple $K = 2$ case of mixed bundling where bids for separate goods and the bundle are submitted. Each method is assessed in terms of revenue optimality, and the effect of introducing optimal reserve prices is analysed. In section 3.2, the optimality of pure bundling over separate-object auctioning is shown. Numeric examples in section 4 with $K = 2$ and uniform values are presented to verify the findings.

2 The Model

The seller has K non-identical objects to sell and there are $N = 2$ potential buyers. Agents i, $i = 0, 1, 2$ is characterised by the vector $\mathbf{v}^i = (v_1^i, v_2^i, ..., v_K^i)$, where v_k^i is agent i's private valuation for object k, and where agent zero denotes the seller. The v_k^i are i.i.d. realisations of $3K$ random variables distributed on an interval $[\underline{v}, \overline{v}] \subseteq \mathbb{R}_0^+$ according to the density $f(\cdot) > 0$. Bidders are assumed to be symmetric in the sense that for all $\mathbf{v}$, $f^1(\mathbf{v}) = f^2(\mathbf{v}) = f(\mathbf{v})$ and $\underline{v}^1 = \underline{v}^2$,

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$\overline{v}^1 = \overline{v}^2$. Let $\mathbf{P}$= $(P^0,\ P^1, P^2)$ be a partition of the set of objects. Assume that bidder i, $i = 1,\ 2$, acquires the subset of objects P^i and makes payment p^i to the seller. The value of objects in P^i are additive, i.e. the objects are neither compliments or substitutes. Thus i's profit is given by $\sum_{k \in P_i} v_k^i$-p^i, and the seller's profit is given by $\sum_{i=1}^{n} p^i - \sum_{k \in K \setminus P^0} v_k^0$.

3 Methods of Auctioning

3.1 Separate-object auctions

We have K separate second-price auctions that proceed simultaneously. The two bidders submit bids for each object. The bids are gathered and the winner pays his or her opponent's bid. Since the value of object k to either bidder is independent of whether other objects are won by that bidder, there is no strategic interaction between each of the auctions. Assume that the seller sets a reserve price, r_k, equal to zero. Then the equilibrium bidding strategy for object k is to submit the bid v_k^i and the auction yields an expected revenue of $E(\sum_{k=1}^{K} min(v_k^1, v_k^2))$. Since bidders are symmetric,

$$\begin{aligned} E(\sum_{k=1}^{K} min(v_k^1, v_k^2)) &= K\, E(min(v_k^1, v_k^2)) \\ &= 2K \int_0^{\overline{v}} v[1 - F(v)]f(v)dv \end{aligned} \tag{1}$$

The objects are allocated efficiently since the winning bidder has the highest value for that object.

For the case where $r_k > 0$, the equilibrium bidding strategy for object k is to submit the bid v_k^i for $v_k^i \geq r_k$, and submit 0 otherwise. The expected seller's revenue for object k, R_k, is given by (Pollard, 2004):

$$R_k = 2 \int_{r_k}^{\overline{v}} F(v)\left(vf(v) - (1 - F(v))\right) dv$$

and the total revenue given by

$$R = 2 \sum_{k=1}^{K} \int_{r_k}^{\overline{v}} F(v)\left(vf(v) - (1 - F(v))\right) dv \tag{2}$$

Since each object's value, v_k, is drawn from the same distribution for both bidders, the optimum $r_k$that maximises (2) will be the same for each object k. Let this optimal reserve price be r. Thus we have

$$R = 2K \int_{r}^{\overline{v}} F(v)\left(vf(v) - (1 - F(v))\right) dv \tag{3}$$

Differentiating (3) shows that the optimal r is given by $r = \frac{1-F(r)}{f(r)}$, which is independent of v_k^0–indeed it is sometimes advantageous to set a reserve price higher than v_k^0. Note that while introducing a reserve price will increase seller's revenue, the auctioning method loses its attractive efficiency. When $r > max(v_k^1, v_k^2) > v_k^0$, the object is not sold thus is not allocated to the agent with highest value.

3.2 Pure bundling auctions

The two bidders submit bids for the bundle containing *all* K objects and the seller allocates the bundle accordingly. Bidder i's value for the bundle is $\sum_{k=1}^{K} v_k^i$ and in the case of no reserve price the equilibrium bidding strategy is to submit this value. Thus the seller's expected revenue is

$$\begin{aligned} R &= E(min(\sum_{k=1}^{K} v_k^1, \sum_{k=1}^{K} v_k^2)) \\ &= 2\int_0^{\overline{v}_R} v[1 - F_R(v)]f_R(v)dv \end{aligned} \tag{4}$$

where $F_R(v) = Pr(\sum_{k=1}^{K} v_k < v)$. $F_R(v)$ is the convolution of K identical distributions, and $\overline{v}_R = K\overline{v}$.

As with separate-object auctioning, when a reserve price is implemented the bidding strategy is to submit bid $\sum_{k=1}^{K} v_k^i$ given this exceeds or equals r and to submit 0 otherwise. Thus the seller's expected revenue is given by

$$R = 2\int_r^{\overline{v}_R} F_R(v)\left(vf_R(v) - (1 - F_R(v))\right)dv \tag{5}$$

The optimal r is given by $r = \frac{1-F_R(r)}{f_R(r)}$.

Now that seller's revenue in both separate and pure bundling has been characterised, the following proposition can be shown to be true.

Proposition 1 *Pure bundling auctions always achieve higher or equal expected revenue than separate-object auctions.*

Proof. For strictly positive realised values $\mathbf{v}^i = (v_1^i, v_2^i, ..., v_K^i)$, it holds that $min(\sum_{k=1}^{K} v_k^1, \sum_{k=1}^{K} v_k^2) \geq \sum_{k=1}^{K} min(v_k^1, v_k^2)$. Thus it also holds that

$$E\left[min(\sum_{k=1}^{K} v_k^1, \sum_{k=1}^{K} v_k^2)\right] \geq E\left[\sum_{k=1}^{K} min(v_k^1, v_k^2)\right]$$

which gives $R_{PB} \geq R_{SA}$.

A more direct proof requires that the following be shown:

$$K\int_0^{\overline{v}} v[1 - F(v)]f(v)dv \geq \int_0^{K\overline{v}} v[1 - F_R(v)]f_R(v)dv$$

With an arbitrary value distribution $F(v)$, explicitly determining an expression for the Kth convolution $F_R(v)$ is very difficult, thus making the approach unfeasible.

3.3 Mixed bundling auctions

The two bidders submit bids for sets of objects as well as for separate objects. The seller collects the bids and determines the winner by finding the optimum allocation that maximises his or her revenue.

The winner determination problem is inherently complex and is NP-complete, meaning that there does not exist a polynomial-time algorithm that is guaranteed to compute the optimal allocation. Worse still, the problem is not what is called uniformly approximable: there does not exist a polynomial-time algorithm and a constant c such that, for all inputs, the algorithm produces and answer that is at least $1/c$ of the correct optimal answer.

To avoid this complexity, we shall consider the simplest case possible: $K = 2$. Bidders submit sealed bids for both objects (A and B) and for the bundle (AB). Bidders are indifferent to whether they win objects A and B separately or win the AB bundle if both outcomes have the same payment.

To determine seller's expected revenue, whether the auction remains incentive compatible when using the Vickery mechanism is necessary knowledge. Fortunately it does remain true that bidding v_k^i is an equilibrium strategy, and a general proof is given in Bikhchandani, et. al. (2001). If bidders bid v_A^i, v_B^i for object A and B respectively and $v_A^i + v_B^i$ for bundle AB, then the auction is simply an exercise of running both separate-object and pure-bundling methods simultaneously. By proposition 1, the auction will always conclude with the bundle being awarded to a single bidder, and the seller's expected revenue will be the same as in pure bundling with no reserve price.

4 Numerical Examples

We consider here an example with $K = 2$ and uniformly distributed independent values.

By (1), the seller's expected revenue for separate object auctioning with no reserve price (R_{SA}) is given by

$$\begin{aligned} R_{SA} &= 4\int_0^1 v - v^2 dv \\ &= 0.67 \end{aligned}$$

When using the optimal reserve price $r = \frac{1}{2}$, the auction yields an expected revenue (R_{SAR}) of

$$\begin{aligned} R_{SAR} &= 4\int_{1/2}^1 v\,(v - 1 + v)\,dv \\ &= 0.83 \end{aligned}$$

By (4), the seller's expected revenue for a pure bundling auction with no reserve price (R_{PB}) is given by

$$R_{PB} = 2\int_0^2 v[1 - F_R(v)]f_R(v)dv$$

$F_R(v) = Pr(v_1 + v_2 < v)$ where v_1, v_2 are random variables and have joint density $f(v_1, v_2) = 1$ in the space $(0, 1) \times (0, 1)$. Using the cdf method gives

$$F_R(v) = \begin{cases} \frac{1}{2}v^2 & 0 < v < 1 \\ 1 - \frac{1}{2}(2-v)^2 & 1 < v < 2 \end{cases}$$

$$f_R(v) = \begin{cases} v & 0 < v < 1 \\ 2 - v & 1 < v < 2 \end{cases}$$

Thus,

$$\begin{aligned} R_{PB} &= 2\int_0^1 v^2[1 - \frac{1}{2}v^2]dv + 2\int_1^2 v[1 - 1 + \frac{1}{2}(2-v)^2](2-v)dv \\ &= 0.77 \end{aligned}$$

The optimal reserve price is given by

$$\begin{aligned} r &= \frac{1 - \frac{1}{2}r^2}{r} \\ \Rightarrow r &= \frac{\sqrt{6}}{3} \end{aligned}$$

Thus with implementing a reserve price the auction yields an expected revenue (R_{SAR}) of

$$\begin{aligned} R_{SAR} &= 2\int_{\frac{\sqrt{6}}{3}}^1 \frac{1}{2}v^2\left(v^2 - (1 - \frac{1}{2}v^2)\right)dv + 2\int_1^2 (1 - \frac{1}{2}(2-v)^2)\left(v(2-v) - (\frac{1}{2}(2-v)^2)\right)dv \\ &= 0.80 \end{aligned}$$

The following table summarises the findings:

Auction format:	Expected revenue:
Separate-object: $r = 0$	0.67
Pure bundling: $r = 0$	0.77
Pure bundling with optimal reserve price: $r = \frac{\sqrt{6}}{3}$	0.80
Separate-object with optimal reserve price: $r = \frac{1}{2}$	0.83

References

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