A Critique of the Arbitrage Pricing Theory

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Abstract

This short note proves that the pricing equation of the Arbitrage Pricing Theory (Ross, 1976) is a mathematical identity. The result of APT is that expected returns on any portfolio are *approximately* linear in factor loadings of K risk factors. We show that the expected return on any portfolio can be expressed *exactly* as the sum of K risk factors where at least two factors are (in-sample) mean-variance efficient portfolios. Such portfolios always exist, hence empirical tests of the APT should always accept the model. It is also shown that if the factors do not exactly equal the return on an efficient portfolio, but are still correlated, the APT also holds approximately we prove a bound on the difference.

1 Introduction

The Arbitrage Pricing Theory (Ross, 1976) is an important result in the financial economics literature. The result of Arbitrage Pricing Theory is that expected returns on any portfolio are *approximately* linear in factor loadings of K risk factors,

$$E[R_i] \approx \alpha + \beta_1\gamma_1 + \beta_2\gamma_2 + ... + \beta_K\gamma_K,$$

where $(\beta_1, ..., \beta_K)$ is a vector of factor loadings for asset i, and $(\gamma_1, ..., \gamma_K)$ is a vector of factor risk premia.

Roll (1977) showed that mean-variance efficiency of a portfolio is a sufficient and necessary condition for the capital asset pricing model (CAPM) to hold. This has been termed the "Roll Critique." For any mean-variance efficient portfolio with expected return $E[R_p]$, then for all assets i,

$$E[R_i] - E[R_{zp}] = \beta_i(E[R_p] - E[R_{zp}])$$

where $E[R_{zp}]$ the unique mean-variance efficient portfolio satisfying $Cov(R_p, R_{zp}) = 0$.

We generalize Roll's result for the APT and prove that the pricing equation of the APT is a mathematical identity. The expected return on any portfolio can be expressed *exactly* as the sum of K risk factors where at least two factors are sample mean-variance efficient portfolios. Given any dataset of returns, there exists an uncountable number of in-sample mean-variance efficient portfolios. Consequently, empirical tests of the APT where the set of possible factors include portfolio returns should always accept the model. An example is the Fama-French SMB and HML factors, which are returns on size and value portfolios respectively. We also show that if two factors are correlated to mean-variance portfolios, the APT also holds approximately and we provide a bound on the difference.

2 The Identity

Suppose there are N risky assets, each earning return $R = (R_i)_{i=1}^N$. Denote the expected return vector $\mu = (ER_i)_{i=1}^N$ and variance-covariance matrix $\Sigma = (E(R_i - \mu_i)(R_j - \mu_j))_{i,j=1}^N$. We do not assume a risk-free asset exists. Let $x = (x_1, ..., x_N)$ denote a vector of portfolio weights, with $\sum_{i=1}^N x_i = 1$. The return on portfolio p is $R_p = x_p^T R$, the expected return is $ER_p = x_p^T \mu$, variance $Var(R_p) = x_p^T \Sigma^V x_p$. A mean-variance efficient portfolio, x_e, solves $x_e = \arg\ \min(x^T \Sigma^V x)$ subject to $x_e = ER_e$ and $x_e^T \mathbf{1} = 1$.

Theorem 1 (Roll's Spanning Result)

For any asset i, and for any n^{th} moment efficient portfolio x_e, we we have the relation

$$E[R_i] = E[R_{ze}] + \beta_i(E[R_e] - E[R_{ze}]), \tag{1}$$

where $\beta_i = \frac{Cov(R_i, R_e)}{Var(R_e)}$ and x_{ze} is any portfolio satisfying $Cov(R_{ze}, R_e) = 0$.

Proof See section A of the appendix.

Immediately we see that (1) is a $K = 1$ version of the APT, where $\alpha = E[R_{zp}]$ and $\gamma = E[R_p] - E[R_{zp}]$. We shall construct a K factor version of (1).

Theorem 2 (APT Identity)

For any any integer $K \geq 1$ and and set of efficient portfolios, $x_{e,k}$, the following relation holds,

$$E[R_i] = \alpha + \beta_1\gamma_1 + \beta_2\gamma_2 + ... + \beta_K\gamma_K,$$

where $\gamma_k = E[R_{e,k} - R_{ze,k}]$, and $x_{ze,k}$ is any portfolio satisfying $Cov(R_{e,k}, R_{ze,k}) = 0$.

Proof See section B of the appendix.

Corollary 1

If a risk-free asset r_f exists, then equation (4) reduces to

$$ER_i - r_f = \frac{1}{K}[\beta_{i,1}(ER_{e,1} - r_f) + \beta_{i,2}(ER_{e,2} - r_f) + ... + \beta_{i,K}(ER_{e,K} - r_f)].$$

This follows immediately since $Cov(r_f, ER_{e,k}) = 0$ for all k (and similarly for higher moments). Beyond this condition, choice of $x_{ze,k}$ was arbitrary.

Theorem 3 (Correlated Factor Identity)

Let $K = 1$ and R_e be the return of an nth moment efficient portfolio. Then for any factor F with return R_F and with $Corr(R_K, R_e) = \rho$, then for all assets i,

$$E[R_i] = \alpha + \beta_{i,F}\gamma_F + O\left(\frac{1}{|\rho|^2} - 1\right).$$

where

$$\beta_{i,F} = \frac{Cov(R_i, R_F)}{Var(R_F)} \text{ for } n = 2, \text{ and similarly for } n = 3,\ n = 4.$$

Proof See section A.3 of the appendix.

Corollary 2 (Approximate APT Identity)

Let $K \geq 1$ and the factors k be correlated to n^{th} moment efficient portfolios with correlation $\rho_k = Corr(R_k, R_{e,k})$. Then

$$E[R_i] = \alpha + \beta_{i,1}\gamma_1 + \beta_{i,2}\gamma_2 + ... + \beta_{i,K}\gamma_K - O\left(\frac{1}{\min_k|\rho_k|^2} - 1\right)$$

where

$$\beta_{i,k} = \frac{Cov(R_i, R_k)}{Var(R_k)}.$$

Corollary 3

Suppose a risk-free asset exists with return r_f. Then:

$$E[R_i] - r_f = \beta_{i,1}\gamma_1 + \beta_{i,2}\gamma_2 + ... + \beta_{i,K}\gamma_K - O\left(\frac{1}{\min_k|\rho_k|^2} - 1\right)$$

where $\beta_{i,k} = \frac{Cov(R_i,R_k)}{Var(R_k)}$ and $\gamma_k = (ER_k - \frac{1}{\rho_k}\sqrt{\frac{Var(R_k)}{Var(R_{e,k})}}(r_f - \zeta))$.

3 Conclusion

We prove that the linear pricing result of the APT can be derived, without assumptions, by Roll (1977) spanning result of any two mean-variance efficient portfolios. This identity holds in any set of asset returns, and therefore empirical tests of the APT should always accept the APT model when portfolio returns are considered for factors. We also derive the APT equation when the factors do no equal efficient portfolios, but are instead correlated. The pricing error depends on the absolute value of correlation and has order $O(\frac{1}{|\rho_k|} - 1)$.

A Appendix

A.1 Proof of Theorem 1

Let x_e denote any mean-variance efficient portfolio other than the minimum variance portfolio. Let $\Sigma = (\sigma_{i,j})_{N\times N}$ denote the matrix of asset covariances. x_e solves the optimization problem

$$x_e = \arg\ \min \frac{1}{2}x^T\Sigma x \text{ subject to } x_e = ER_e,\ \ x_e^T\mathbf{1} = 1.$$

The necessary and sufficient first order conditions for a solution are:

$$\begin{aligned} \Sigma x_e &= \lambda\mu + \kappa\mathbf{1} \\ ER_e &= x_e^{'}\mu \\ 1 &= x_e^{'}\mathbf{1}. \end{aligned} \tag{2}$$

Solving these FOCs for x_e' yields

$$x_e = (\lambda\Sigma^{-1}\mu + \kappa\Sigma^{-1}1),\ \ x_e'\mu = E[R_e]$$

where λ and κ are

$$\lambda = \frac{1}{D}(CE[R_p] - A),\ \ \gamma = \frac{1}{D}\left(B - AE[R_p]\right). \tag{3}$$

and where the scalars (A, B, C, D) are defined as

$$A = \mu^{'}\Sigma^{-1}\mathbf{1}, \;\; B = \mu^{'}\Sigma^{-1}\mu, \;\; C = \mathbf{1}\Sigma^{-1}\mathbf{1}, \;\; D = BC - A^2.$$

By definition of covariance,

$$Cov(R_p, R_q) = x_p'\Sigma x_q.$$

for all portfolios x_p and x_q. Suppose x_p is efficient. Using the first order (2) it follows that

$$\begin{aligned} Cov(R_p, R_q) &= (\lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}\mathbf{1})'\Sigma x_q \\ &= \lambda\mu'\Sigma^{-1}\Sigma x_q + \gamma 1'\Sigma^{-1}x_q \\ &= \lambda\mu^{'}x_q + \gamma 1' x_q \\ &= \lambda E[R_q] + \gamma. \end{aligned}$$

Substituting in values of λ and γ yields the expression

$$Cov(R_p, R_q) = \frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}.$$

Let x_{zp} denote any portfolio satisfying $Cov(R_p, R_{zp}) = 0$. Such a portfolio always exists and lies on the frontier. The expected return $E[R_{zp}]$ is given by rearranging the covariance expression:

$$E[R_{zp}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}}.$$

The variance of R_p is $Cov(R_p, R_p)$. Dividing the above by $Var(R_p)$ yields

$$\frac{Cov(R_p, R_q)}{Var(R_p)} = \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)\left(E[R_q] - \frac{A}{C}\right) + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}$$

Isolating $E[R_q]$ and setting $\frac{Cov(R_p,R_q)}{Var(R_p)} = \beta_q$,

$$\begin{aligned} E[R_q] &= \frac{\beta_q Var(R_p) - \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + \frac{A}{C} - \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}} \\ &= \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} + E[R_{zp}]. \end{aligned}$$

Working on the first term yields:

$$\begin{aligned} \frac{\beta_q Var(R_p)}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} &= \beta_q \frac{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)^2 + \frac{1}{C}}{\frac{C}{D}\left(E[R_p] - \frac{A}{C}\right)} \\ &= \beta_q\left(E[R_p] - \frac{A}{C} + \frac{\frac{D}{C^2}}{E[R_p] - \frac{A}{C}}\right) \\ &= \beta_q(E[R_p] - E[R_{zp}]) \end{aligned}$$

Substituting this this back in to , we get

$$E[R_q] = \beta_q(E[R_p] - E[R_{zp}]) + E[R_{zp}].$$

A.2 Proof of Theorem 2

Pick any K efficient portfolios; (1) holds true for each one, so summing over K yields

$$KE[R_i] = \sum_{k=1}^{K} E[R_{zp_k}] + \sum_{k=1}^{K} \beta_{i,k}(E[R_{p_k}] - E[R_{zp_k}])$$

Divide by K:

$$E[R_i] = \frac{1}{K}\sum_{k=1}^{K} E[R_{zp_k}] + \sum_{k=1}^{K} \beta_{i,k}\frac{E[R_{p_k}] - E[R_{zp_k}]}{K}$$

and setting $\alpha = \frac{1}{K}\sum_{k=1}^{K} E[R_{zp_k}]$, $(E[R_{p_k}] - E[R_{zp_k}]) = K\gamma_i$ gives us

$$E[R_i] = \alpha + \beta_1\gamma_1 + \beta_2\gamma_2 + ... + \beta_K\gamma_K \tag{4}$$

where the factor loadings reflect the normalised covariance between R_i and market factor R_{p_k}.

A.3 Proof of Theorem 3

By Theorem 1, we have

$$E[R_i] = \alpha + \beta_1\gamma_1$$

where $\beta_i\gamma_1 = \frac{Cov(R_i,R_{e,1})}{Var(R_{e,1})}[ER_e - ER_{ze}]$. Since $Corr(R_K, R_e) = \rho$, we have

$$R_e = \zeta + \xi R_F + \varepsilon$$

where $E\varepsilon = 0$, $Var(\varepsilon) = 9\xi\left(\frac{1}{|\rho|^2} - 1\right)$, and $Cov(R_F, \varepsilon) = 0$; see appendix for proof. Plugging this in yields\

$$\begin{aligned}
\beta_i\gamma_1 &= \frac{\xi Cov(R_i, R_F) + Cov(R_i, \varepsilon)}{\xi^2 Var(R_F) + Var(\varepsilon)}[\zeta + \xi ER_F - ER_{ze}] \\
&= \frac{Cov(R_i, \varepsilon)}{Var(R_{e,1})}[ER_{e,1} - ER_{Ze,1}] + \frac{Cov(R_i, R_F)}{Var(R_F) + Var(\varepsilon)/\xi^2}[ER_F - \frac{1}{\xi}(ER_{ze} - \zeta)] \\
&= \frac{Cov(R_i, \varepsilon)}{Var(R_{e,1})}[ER_{e,1} - ER_{Ze,1}] + \frac{Cov(R_i, R_F)}{Var(R_F) + 9\left(\frac{1}{|\rho|^2} - 1\right)}[ER_F - \frac{1}{\xi}(ER_{ze} - \zeta)]
\end{aligned}$$

Using the fact that $\frac{a}{b+c} = \frac{a}{b} - \frac{ac}{b(b+c)}$,

$$\begin{aligned}
\frac{Cov(R_i, R_F)}{Var(R_F) + 9\left(\frac{1}{|\rho|^2} - 1\right)} &= \frac{Cov(R_i, R_F)}{Var(R_F)} - \frac{Cov(R_i, R_F)\left(\frac{1}{|\rho|^2} - 1\right)}{Var(R_F)^2 + 9Var(R_F)\left(\frac{1}{|\rho|^2} - 1\right)} \\
&= \beta_{i,F} - O\left(\frac{1}{|\rho|^2} - 1\right)
\end{aligned}$$

so

$$\beta_{1,i}\gamma_1 = \frac{Cov(R_i, \varepsilon)}{Var(R_{e,1})}[ER_{e,1} - ER_{Ze,1}] + \beta_{i,F}[ER_F - \frac{1}{\xi}(ER_{ze} - \zeta)] - O\left(\frac{1}{|\rho|^2} - 1\right)$$

If we assume $Cov(R_i, \varepsilon) = 0$, then we have

$$E[R_i] = \alpha + \beta_{i,F}\gamma_F - O\left(\frac{1}{|\rho|^2} - 1\right).$$

where $\beta_{i,F} = \frac{Cov(R_i,R_F)}{Var(R_F)}$, $\gamma_F = ER_F - \frac{1}{\xi}(ER_{ze} - \zeta)$.

References

[1] Ross, S., "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341-360, 1976.

[2] Roll, R., "A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory," *Journal of Financial Economics,* 4,2, 129-176, 1977.